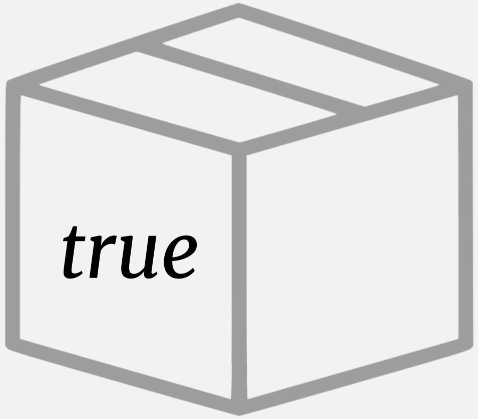
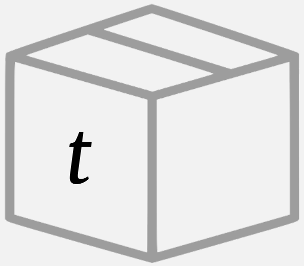
Boxes and Locks

Nachiappan V.

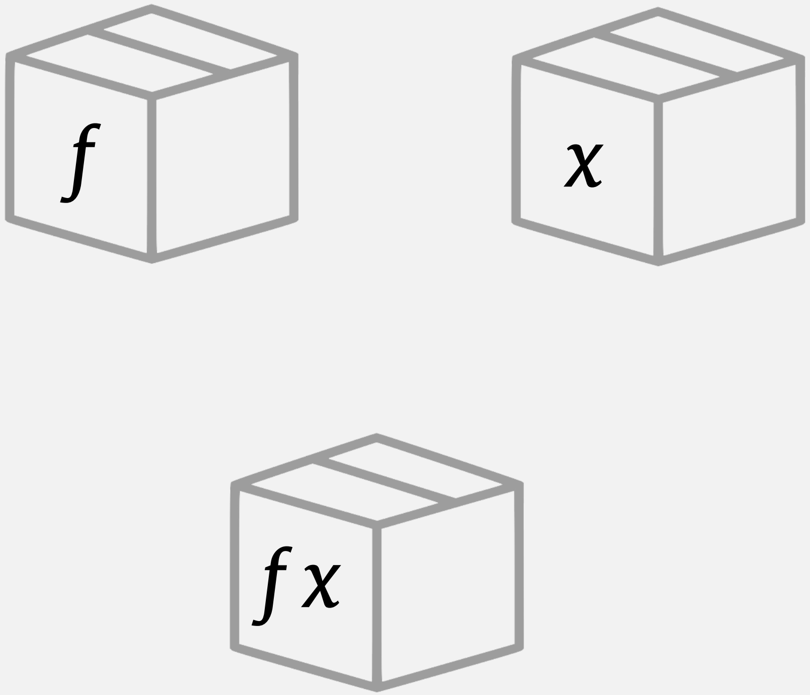


$: \quad \Box \mathit{Bool}$

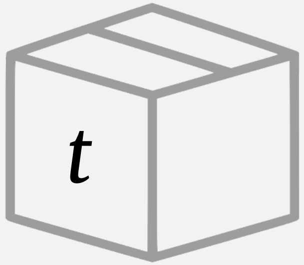
$\vdash t : A$

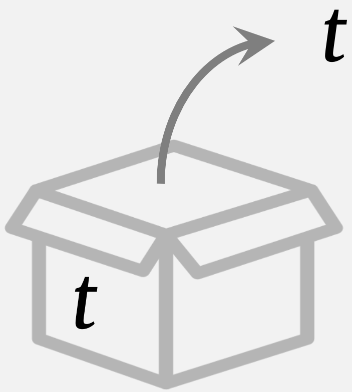


$$\frac{\vdash A}{\Gamma \vdash \Box A}\ \textsc{Nec}.$$

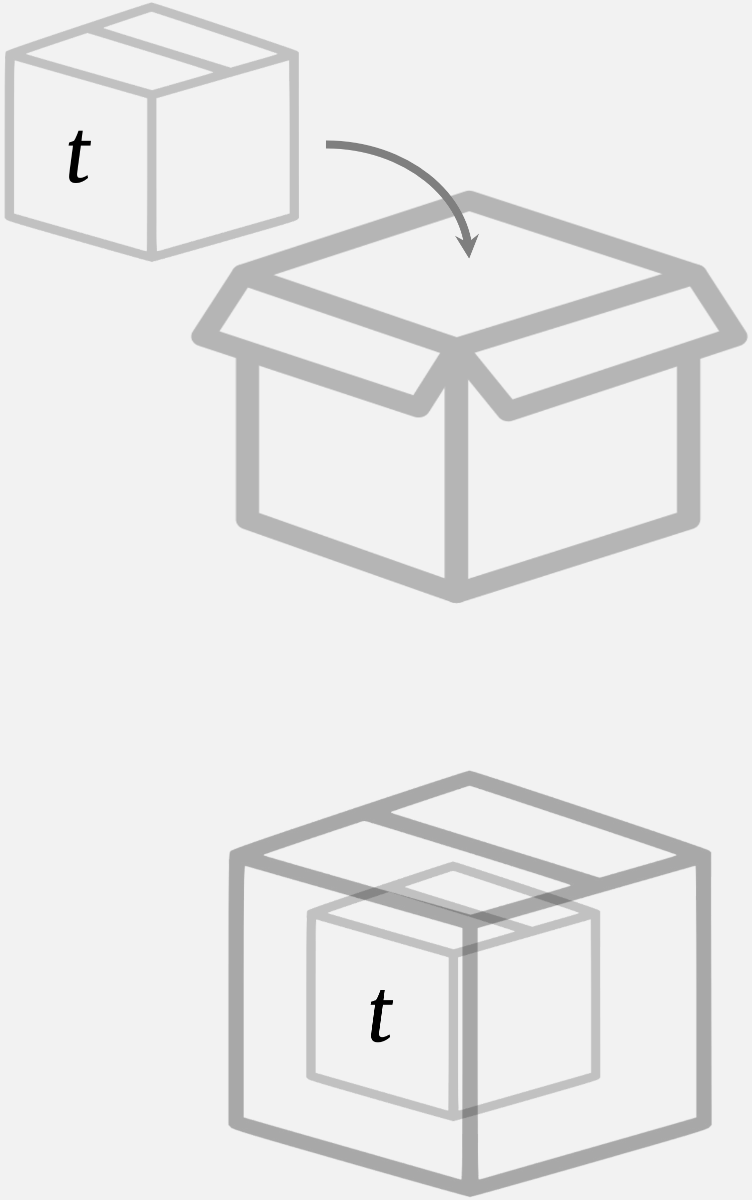


$$\mathrm{K}\colon \Gamma \vdash \Box(A \to B) \to \Box A \to \Box B$$





$$\mathrm{T}\colon \Gamma \vdash \Box A \rightarrow A$$



$$4\colon\ \Gamma \vdash \Box A \to \Box\Box A$$

IS4

K: $\Gamma \vdash \Box(A \to B) \to \Box A \to \Box B$

Today's focus

T: $\Gamma \vdash \Box A \to A$

4: $\Gamma \vdash \Box A \to \Box\Box A$

IK4

R: $\Gamma \vdash A \to \Box A$

IR

Fitch-style IK

$\Gamma ::= \cdot \mid \Gamma, x : A \mid \Gamma, \text{🔒}$

$$\frac{}{\Gamma, x : A, \Gamma' \vdash x : A} \; \text{🔒} \notin \Gamma'$$

$$\frac{\Gamma \vdash t : \Box A}{\Gamma, \text{🔒}, \Gamma' \vdash \mathbf{unbox}\ t : A} \; \text{🔒} \notin \Gamma' \qquad \frac{\Gamma, \text{🔒} \vdash t : A}{\Gamma \vdash \mathbf{box}\ t : \Box A}$$

Axiom K is derivable

$$\dfrac{\dfrac{\ldots, \text{🔒} \vdash \textbf{unbox}\ f : A \to B \qquad \ldots, \text{🔒} \vdash \textbf{unbox}\ x : A}{\Gamma, f : \Box(A \to B), x : \Box A, \text{🔒} \vdash \textbf{app}\ (\textbf{unbox}\ f)\ (\textbf{unbox}\ x) : B}}{\Gamma, f : \Box(A \to B), x : \Box A \vdash \textbf{box}\ (\textbf{app}\ (\textbf{unbox}\ f)\ (\textbf{unbox}\ x)) : \Box B}$$

$$\dfrac{\Gamma \vdash t : \Box A}{\Gamma, \text{🔒}, \Gamma' \vdash \textbf{unbox}\ t : A}\ \text{🔒} \notin \Gamma' \qquad \dfrac{\Gamma, \text{🔒} \vdash t : A}{\Gamma \vdash \textbf{box}\ t : \Box A}$$

Weakening Rules

$\Gamma ::= \cdot \mid \Gamma, x : A \mid \Gamma, 🔒$

$$\frac{}{\textbf{base} : \Gamma \leq []}$$

$$\frac{w : \Gamma' \leq \Gamma}{\textbf{drop}\ w : \Gamma', A \leq \Gamma}$$

$$\frac{w : \Gamma' \leq \Gamma}{\textbf{keep}\ w : \Gamma', A \leq \Gamma, A}$$

$$\frac{w : \Gamma' \leq \Gamma}{\textbf{keep-🔒}\ w : \Gamma', 🔒 \leq \Gamma, 🔒}$$

Necessitation is admissible

$$\vdash A \quad \Longrightarrow \quad \vdash \Box A$$

$$\dfrac{\dfrac{\vdash t : A}{\blacksquare \vdash t : A}\ \text{Weaken with } \textbf{base}}{\vdash \mathbf{box}\ t : \Box A}$$

What about ”Denecessitation”?

$$\vdash \Box A \quad \Longrightarrow \quad \vdash A$$

How would you prove this?

$$\frac{\Gamma \vdash t : \Box A}{\Gamma, \text{🔒}, \Gamma' \vdash \mathbf{unbox}\ t : A}\ \text{🔒} \notin \Gamma' \qquad \frac{\Gamma, \text{🔒} \vdash t : A}{\Gamma \vdash \mathbf{box}\ t : \Box A}$$

Normalization Rules

$$(\beta) \quad \textbf{unbox}\ (\textbf{box}\ t) \ \longrightarrow\ t$$

$$(\eta) \quad \Gamma \vdash t \ \longrightarrow\ \textbf{box}\ (\textbf{unbox}\ t) : \Box A$$

Normal Forms

$$\frac{\Gamma, \text{🔒} \vdash_{\mathrm{nf}} t : A}{\Gamma \vdash_{\mathrm{nf}} \mathbf{box}\ t : \Box A}$$

$$\frac{\Gamma \vdash_{\mathrm{ne}} t : \Box A}{\Gamma, \text{🔒}, \Gamma' \vdash_{\mathrm{ne}} \mathbf{unbox}\ t : A}\ \text{🔒} \notin \Gamma'$$

Normalization by Evaluation

$$\llparenthesis \_ \rrparenthesis \ : (\Gamma \vdash A) \to \llbracket \Gamma \rrbracket_\Delta \to \llbracket A \rrbracket_\Delta$$

$$\downarrow_A \ : \llbracket A \rrbracket_\Gamma \to (\Gamma \vdash_{\mathrm{nf}} A)$$

$$norm\ t = \downarrow (\llparenthesis\, t\, \rrparenthesis\ \gamma_{id})$$

Interpretation

$$[\![\tau]\!]_\Gamma = \Gamma \vdash_{\mathrm{nf}} \tau$$

$$[\![A \to B]\!]_\Gamma = \Gamma' \leq \Gamma \to [\![A]\!]_{\Gamma'} \to [\![B]\!]_{\Gamma'}$$

$$\boxed{[\![\Box A]\!]_\Gamma = \mathrm{Box}_\Gamma\ [\![A]\!]}$$

$$[\![\cdot]\!]_\Gamma = \top$$

$$[\![\Delta, A]\!]_\Gamma = [\![\Delta]\!]_\Gamma \times [\![A]\!]_\Gamma$$

$$\boxed{[\![\Delta, 🔒]\!]_\Gamma = \mathrm{Lock}_\Gamma\ [\![\Delta]\!]}$$

Boxes and Locks

$$\mathcal{A} : \mathrm{Ctx} \to \mathrm{Set}$$

$$\frac{x : \mathcal{A}_{\Gamma,\text{🔒}}}{\mathsf{box}\ x : \mathrm{Box}_{\Gamma}\ \mathcal{A}} \qquad \frac{x : \mathcal{A}_{\Gamma}}{\mathsf{lock}\ x : \mathrm{Lock}_{\Gamma,\text{🔒},\Gamma'}\ \mathcal{A}}\ \text{🔒} \notin \Gamma'$$

$$(\text{🔒} \notin \Gamma')$$

$$unbox : \mathrm{Box}_{\Gamma}\ [\![\ A\ ]\!] \to [\![\ A\ ]\!]_{\Gamma,\text{🔒},\Gamma'}$$

$$unbox\ (\mathsf{box}\ x) = wk\ x$$

Evaluation

$$
\begin{aligned}
(\!|\ \_\ |\!) \;&: (\Gamma \vdash A) \to [\![\Gamma]\!]_\Delta \to [\![A]\!]_\Delta \\
(\!|\ \mathbf{box}\ t\ |\!)\ \gamma &= \mathsf{box}\ ((\!|\ t\ |\!)\ \gamma) \\
(\!|\ \mathbf{unbox}\ t\ |\!)\ \langle \gamma, \_ \rangle &= (\!|\ \mathbf{unbox}\ t\ |\!)\ \gamma \\
(\!|\ \mathbf{unbox}\ t\ |\!)\ (\mathsf{lock}\ \gamma) &= \mathit{unbox}\ ((\!|\ t\ |\!)\ \gamma)
\end{aligned}
$$

$$
\frac{\Gamma \vdash t : \Box A}{\Gamma, 🔒, \Gamma' \vdash \mathbf{unbox}\ t : A}\ 🔒 \notin \Gamma' \qquad \frac{\Gamma, 🔒 \vdash t : A}{\Gamma \vdash \mathbf{box}\ t : \Box A}
$$

Reification and Reflection

$$\downarrow_A \;:\; [\![A]\!]_\Gamma \to (\Gamma \vdash_{\mathrm{nf}} A)$$

$$\downarrow_{\square A}\;(\mathsf{box}\; x) \quad = \mathbf{box}\;(\downarrow_A\;\; x)$$

$$\uparrow_A \;:\; (\Gamma \vdash_{\mathrm{ne}} A) \to [\![A]\!]_\Gamma$$

$$\uparrow_{\square A}\; m \qquad\qquad = \mathsf{box}\;(\uparrow_A\;(\mathbf{unbox}\; m))$$

Denecessitation can be proved using normal forms

$$\vdash \Box A \Longrightarrow \vdash A$$

$$\dfrac{\dfrac{\times}{\vdash_{\mathrm{ne}} t : \Box B}}{\blacksquare \vdash_{\mathrm{ne}} \mathbf{unbox}\ t : B}$$

$$\vdots$$

$$\dfrac{\blacksquare \vdash_{\mathrm{nf}} t : A}{\vdash_{\mathrm{nf}} \mathbf{box}\ t : \Box A}$$

No neutrals in empty context, dismissed!

Only culprit that introduces 🔒

Remove 🔒 to get $\vdash_{\mathrm{nf}} t : \Box A$

Why?

1. Useful

- Towards confluence, decidability, consistency, etc.

- Simplify / prove new application-specific theorems?

  (e.g., partial evaluation/staging theorems, noninterference, etc.)

- Type-directed partial evaluation for modal type systems

2. Extensible: Boxes, Locks and (fake) Diamonds!

IK$\blacklozenge$

$$\frac{\Gamma \vdash t : A}{\Gamma, \text{🔒}, \Gamma' \vdash \mathbf{dia}\ t : \blacklozenge A}\ \text{🔒} \notin \Gamma' \qquad \frac{\Gamma \vdash t : \blacklozenge A \qquad A, \text{🔒} \vdash u : B}{\Gamma \vdash \mathbf{bind}\ t\ u : B}$$

Interpreting $\blacklozenge$

$$\frac{x : \mathrm{Lock}_{\Gamma} \mathcal{A}}{\mathsf{val}\ x : \mathrm{Dia}_{\Gamma}\ \mathcal{A}} \qquad \frac{m : \Gamma \vdash_{\mathrm{ne}} \blacklozenge A \qquad y : \mathrm{Dia}_{A, \text{🔒}}\ \mathcal{A}}{\mathsf{bindst}\ m\ y : \mathrm{Dia}_{\Gamma}\ \mathcal{A}}\ \text{🔒} \notin \Gamma'$$

2. Extensible: IS4, IK4, IR

IS4

$$\text{K: } \Gamma \vdash \Box(A \to B) \to \Box A \to \Box B$$

$$\text{T: } \Gamma \vdash \Box A \to A$$

$$\text{4: } \Gamma \vdash \Box A \to \Box\Box A$$

IK4

$$\text{R: } \Gamma \vdash A \to \Box A$$

IR

3. Comprehensible: Reduction traces (Demo)

```
η : Tm Γ a → Tm Γ (□ ◆ a)
η t = box (dia t ')

ε : Tm Γ (◆ □ a) → Tm Γ a
ε t = bind t (unbox x0 ')

fmap : Tm Γ a → Tm (Γ 🔒) (◆ a)
fmap t = dia t '

zigR : Tm (Γ `, a 🔒) (◆ a)
zigR = ε (fmap (η x0))
-- bind (dia (box (dia x0))) (unbox x0)
```

```
pattern x0 = var ze
```

De Bruijn Index 0

Normalizing zigR

```
U:---  KDemo.agda     36% L36     (Agda:Checked +4)
dia x0 '
```

Printing reduction trace for zigR

```
U:---  KDemo.agda     44% L43     (Agda:Checked +4)
β◆ ◅ β□ ◅ done
```

4. Elegant: Boxes and Locks, in an Adjoint relationship

$$unit : \mathcal{A} \overset{\cdot}{\longrightarrow} \mathrm{Box}\ (\mathrm{Lock}\ \mathcal{A}) \qquad\qquad counit : \mathrm{Lock}\ (\mathrm{Box}\ \mathcal{A}) \overset{\cdot}{\longrightarrow} \mathcal{A}$$

$$unit\ x = \mathsf{box}\ (\mathsf{lock}\ x) \qquad\qquad counit\ (\mathsf{lock}\ (\mathsf{box}\ x)) = wk\ x$$

$$\begin{tikzcd}
\mathrm{Lock}\ \mathcal{A} \arrow[r, "id"] & \mathrm{Lock}\ \mathcal{A} \arrow[dl, "\mathrm{Lock}_1(unit)"] \arrow[l, phantom, shift right=6ex, "\circlearrowright"] \\
\mathrm{Lock}\ (\mathrm{Box}\ (\mathrm{Lock}\ \mathcal{A})) \arrow[u, "counit"] &
\end{tikzcd}$$

What would you like to see more of?

Modalities (S4, K4, R, etc.)

Semantics

Applications

Extensions (0, +, Nat, etc.)

EOM