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Cover Semantics for Intuitionistic Modalities

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Abstract

Intuitionistic modal logic (IML) has inspired several developments in programming languages including modal type systems for staging, computational effects and language-based security. IMLs are typically studied using Kripke-style relational semantics, which simplifies proofs of meta-theoretic properties, such as completeness and consistency, by making it easy to construct models. Kripke-style relational semantics, however, relies upon classical reasoning principles, which makes it unappealing from a computational perspective and unsuitable for formalization in a constructive type theory. Goldblatt provides an alternative semantics for IMLs by extending Beth-Kripke-Joyal-style "cover" semantics for intuitionistic propositional logic with relations to support modalities. Goldblatt's "relational cover" semantics overcomes classical reasoning but introduces a new limitation: it relies upon a "modal localization" condition that restricts the class of models and complicates model construction. Goldblatt bypasses this restriction by using intricate order-theoretic completion arguments to prove completeness. In this article, we present a conservative extension of relational cover semantics that alleviates this restriction and is amenable to simpler and standard model construction techniques. We formalize our semantics in Agda and prove completeness constructively in the style of Normalization by Evaluation for a variety of IMLs featuring independent box and diamond modalities.

Keywords: constructive completeness, intuitionistic modal logic, normalization by evaluation

1 Introduction

Intuitionistic modal logic (IML) is the study of formal logics that extend intuitionistic propositional logic with modalities such as the box ($\Box$) and diamond ($\Diamond$) connectives. Early work on IML can be found beginning with Fitch [26] in the late 1940s, followed by pioneering contributions from Fischer-Servi [25,43], Božić and Došen [14], Sotirov [45], and many others since [42,48,44]. These studies have found various applications in computer science, notably inspiring the design of modal type systems in programming languages for distributed computing [38], meta-programming [21,39], guarded recursion [11,10] and language-based security [27,12,1]. Several recent developments [46,30,36,31,3] in modal type systems can be directly traced back to earlier work [13,41,40] on the proof theory and natural deduction calculi for IMLs.

In contrast to the enthusiastic adoption of the proof theory for IMLs, the model-theoretic strengths of IMLs remain largely under-utilized in the study of programming languages. This is an opportunity missed: modal logics enjoy a rich semantic foundation with slick model construction techniques that could simplify the way we currently reason about modal type systems. The trouble, however, lies in the fact that most developments in the semantics of IMLs rely upon classical reasoning principles, such as proof by contradiction or the axiom of choice, which inhibits their adoption in the study of programming languages. The objective of this article is to develop a new semantics for IMLs that does not require classical reasoning.

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Kripke-style relational semantics. The standard semantics used to model IMLs extends Kripke's semantics for IPL [35] using an *accessibility* relation [44]. The truth of a formula is given using a triple $F = (W, \sqsubseteq, R)$ known as a *frame*, which consists of a set W of worlds, a partial order relation $\sqsubseteq$ on worlds and an accessibility relation $R \subseteq W \times W$ subject to certain compatibility conditions. Given a model $\mathcal{M} = (F, V)$, consisting of a frame F and a valuation V of propositional atoms, we say that a formula A is true for a world w whenever the *satisfaction* relation $\mathcal{M}, w \Vdash A$ holds. The satisfaction relation is defined for an IML by extending the usual definition for IPL originally given by Kripke [35]. In particular, satisfaction is defined for the *positive*, i.e. falsity ($\bot$) and disjunction ($\vee$), connectives as:

$$\mathcal{M}, w \Vdash \bot \text{ iff false} \qquad\qquad \mathcal{M}, w \Vdash A \vee B \text{ iff } \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B$$

The satisfaction of modal formulas, typically $\Box A$ and $\Diamond A$, is defined using the accessibility relation R and can vary significantly depending on the logic and applications under consideration. A comprehensive formal analysis of these variations can be found in a recent survey of IMLs by De Groot et al. [23], who propose a sweeping generalization of several common variants for boxes and diamonds as:

$$\mathcal{M}, w \Vdash \Box A \text{ iff } \forall w'.\, w \sqsubseteq w' \text{ implies } \forall v.\, w' \mathrel{R} v \text{ implies } \mathcal{M}, v \Vdash A$$
$$\mathcal{M}, w \Vdash \Diamond A \text{ iff } \forall w'.\, w \sqsubseteq w' \text{ implies } \exists v.\, w' \mathrel{R} v \text{ and } \mathcal{M}, v \Vdash A$$

Kripke-style relational semantics has been used to model a wide variety of IMLs for their remarkable ability to simplify proofs of complex meta-theoretic properties such as completeness and consistency by making it easy to construct a model. To construct a model, we need to identify four parameters $(W, \sqsubseteq, R, V)$ and show that they satisfy the necessary compatibility conditions—a process that requires far less ingenuity, for example, in comparison with algebraic models based on Heyting algebras. The price to pay, however, is that the proof of completeness, in particular the canonical model construction, relies upon the existence of *prime filters* that presumes availability of the axiom of choice [33, Remark 2.2].

Relational cover semantics. Goldblatt [28,29] provides an alternative semantics for IMLs by extending a re-development of the so-called cover, or "Kripke-Joyal", semantics for IPL using an accessibility relation for each modality in an IML. Goldblatt's re-development presents ideas that "originated in topos theory, in the logic of categories of sheaves", typically attributed to Joyal [34, Section 1] and Beth [8], "in a more general context that abstracts away from topological spaces". A detailed account of this re-development can be found in [29, Section 3]. Instead of a frame, the truth of a formula is given using a *cover system* $C = (W, \sqsubseteq, \lhd)$, consisting of a partial order $(W, \sqsubseteq)$ and a *covering* relation $\lhd \subseteq W \times \mathcal{P}(W)$, accompanied by accessibility relations such as $R_\Box$ and $R_\Diamond$ subject to certain compatibility conditions.

The definition of the satisfaction relation departs notably from Kripke-style semantics for *both* the positive connectives and the modalities. The clauses below define satisfaction for the positive connectives in relational cover semantics by removing the "immediacy" that Kripke-style semantics necessitates.

$$\mathcal{M}, w \Vdash \bot \quad \text{iff } w \lhd \emptyset$$
$$\mathcal{M}, w \Vdash A \vee B \text{ iff } \exists \alpha.\, w \lhd \alpha \text{ and } \forall v \in \alpha.\, \mathcal{M}, v \Vdash A \text{ or } \mathcal{M}, v \Vdash B$$

A formula $A \vee B$ is true for world w iff either A or B is true, not necessarily for w itself, but for all worlds v in some subset $\alpha \subseteq W$ that *covers* w. Similarly, $\bot$ is true for a world w iff the empty set $\emptyset$ covers w.

The clauses below define satisfaction for the $\Box$ and $\Diamond$ modalities in a somewhat unusual manner, in contrast to Kripke-style semantics, by treating both modalities like diamonds in classical modal logic.

$$\mathcal{M}, w \Vdash \Box A \text{ iff } \exists v.\, w \mathrel{R_\Box} v \text{ and } \mathcal{M}, v \Vdash A \qquad\qquad \mathcal{M}, w \Vdash \Diamond A \text{ iff } \exists v.\, w \mathrel{R_\Diamond} v \text{ and } \mathcal{M}, v \Vdash A$$

Goldblatt treats all modalities alike under the slogan that "there is more to intuitionistic modal logic than the generalisation of properties of boxes and diamonds from Boolean modal logic" [28]. The logical properties of each individual modality is modeled by imposing additional conditions on its respective accessibility relation. For example, the necessitation rule for the box modality (if A is a valid formula, then so is $\Box A$) is modeled by requiring $R_\Box$ to be a serial relation [28, Section 3]. The result is a uniform semantics that models a variety of IMLs including Bellin et al.'s Constructive K (CK) [6], Bierman and de

Paiva's Constructive S4 (CS4) [9] and Fairtlough and Mendler's Propositional Lax Logic (PLL) [24]. These results readily extend further to weaker IMLs including sublogics of CK, namely Božić and Došen's $CK_\Box$ and $CK_\Diamond$ [14], and sublogics of PLL, namely the logics SL, SRL and SJL [47].

A notable character of relational cover semantics is that it does not demand classical reasoning since Goldblatt's completeness proofs do not use prime filters. Relational cover semantics, however, introduces a new problem: it no longer supports standard model construction techniques used to construct canonical models. In an attempt to prove completeness by constructing a Henkin-style canonical model, Goldblatt encounters a "stumbling block" [28, Section 8] due to a condition imposed on relational cover models known as *modal localization*, which has to do with an interaction between the relations R and $\lhd$. Goldblatt bypasses this roadblock by resorting to the use of *MacNeille completion* to construct a different kind of model that satisfies modal localization. The details of Goldblatt's construction are rather intricate and more involved than well-known techniques used to prove completeness for the cover semantics of IPL—reasons which have likely inhibited the larger adoption of relational cover semantics in IML literature.

Two-cover semantics. In this article, we present a conservative extension of relational cover semantics, called *two-cover semantics*, by replacing the accessibility relation $R \subseteq W \times W$ that accompanies a cover system in a relational cover model with a *modal covering* relation $\blacktriangleleft \subseteq W \times \mathcal{P}(W)$. For a world w, the modal covering relation relaxes the concept of a possible "future" world $v \in W$, given by the relationship $w \; R \; v$, to a collection or *neighborhood* of future worlds $\alpha \subseteq W$, given by the relationship $w \blacktriangleleft \alpha$. The resulting semantics retains the convenience of model construction in Kripke-style relational semantics while continuing to avoid classical reasoning as in relational cover semantics. We show that two-cover semantics can be used to model four important IMLs, which are namely:

(i) a minimal monotone modal logic CM featuring a modality $\heartsuit$ that generalizes both $\Box$ and $\Diamond$ and exhibits only the monotonicity rule (if the formula $A \Rightarrow B$ is valid, then so is $\heartsuit A \Rightarrow \heartsuit B$)

(ii) the minimal lax logic SL with a modality $\Diamond$ that exhibits only the axiom $\text{S} : A \wedge \Diamond B \Rightarrow \Diamond(A \wedge B)$

(iii) the full lax logic PLL that extends SL with axioms $\text{R} : A \Rightarrow \Diamond A$ and $\text{J} : \Diamond\Diamond A \Rightarrow \Diamond A$

(iv) the minimal box logic $CK_\Box$ with a modality $\Box$ that exhibits the necessitation rule (if A is valid, then so is $\Box A$) and the distribution axiom $\text{K} : \Box(A \Rightarrow B) \Rightarrow \Box A \Rightarrow \Box B$

The logic CM can be found in a recent study of monotone logics by De Groot [22, Definition 2.9], where we take a single monotone modality $\heartsuit$ instead of $\Box$ and $\Diamond$. CM is the logic of *Modal Heyting algebras* [28, Section 4] and the monotonicity rule in CM corresponds to functoriality in category theory. For our purposes, CM serves as a small toy logic with a non-trivial extension to IPL that makes it easy to illustrate the main ideas underlying two-cover semantics. The logic SL (for "S-lax" logic) is a minimal sublogic of PLL, the latter of which has been studied extensively [4,7,24] and is well known as the IML corresponding to strong monads [37]. The axiom S corresponds to strength of a functor, while the axioms R and J correspond to the properties of a monad. The logic $CK_\Box$ is the smallest box-only IML that underlies the most widely studied box-only IML $CS4_\Box$. $CK_\Box$ was given a dual-context natural deduction system by Kavvos [32] by following the influential work of Pfenning and Davies [40] on $CS4_\Box$. For our purposes, $CK_\Box$ serves as a example of an IML which can be modeled using two-cover semantics despite requiring a special proof system that departs from the usual single-context systems used for the other logics.

We prove soundness for these logics by showing that two-cover models determine equivalent algebraic models, and prove completeness constructively in the style of Normalization by Evaluation [18,16,17] by constructing a Henkin-style canonical model. Furthermore, we show that the completeness proofs can be readily refined to give a normalization algorithm that normalizes proofs in the respective natural deduction system of the logic, which yields as corollaries the subformula property and logical consistency. All theorems in this article have been formalized in Agda, and the formalization can be found at the URL:

https://github.com/nachivpn/cover

2 Overview of Cover Semantics

In this section, we begin with a recap of cover semantics for IPL (Section 2.1) and give an overview of the trouble with relational cover semantics (Section 2.2). We then illustrate our new semantics by giving two-cover semantics for the logic CM (Section 2.3) and extend this to the remaining IMLs in the upcoming sections. The results in sections Sections 2.1 and 2.2 are well-known and due to Goldblatt [28].

2.1 Cover Models for IPL

The language of IPL consists of formulas defined inductively by propositional atoms (p, q, r, etc.), constants $\top$ and $\bot$, and binary logical connectives $\wedge$, $\vee$ and $\Rightarrow$. As usual, the connectives $\wedge$ and $\vee$ have higher operator precedence than $\Rightarrow$, and all binary connectives associate to the right when they are nested.

$$\textsf{Prop}\quad A, B := p, q, r, \ldots \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \Rightarrow B \qquad \textsf{Ctx}\quad \Gamma, \Delta := \cdot \mid \Gamma, A$$

The constants $\top$ and $\bot$ respectively denote universal truth and falsity, and the connectives $\wedge$, $\vee$ and $\Rightarrow$ respectively denote conjunction, disjunction and implication. A context Γ is a finite multiset of formulas $A_1, A_2, ..., A_n$, and $\cdot$ denotes the empty context. A sequent-style natural deduction proof system for IPL is given using the *inference* rules defined in Figure 1. A *judgment* $\Gamma \vdash A$ is an assertion that denotes formula A has a proof under the assumption that all formulas in context Γ have a proof. A judgment $\Gamma \vdash A$ *holds*, written simply as "$\Gamma \vdash A$", when it can be derived using the inference rules in Figure 1.

$$\textsc{Hyp}\ \frac{A \in \Gamma}{\Gamma \vdash A} \qquad \top\textsc{-Intro}\ \ \Gamma \vdash \top \qquad \bot\textsc{-Elim}\ \frac{\Gamma \vdash \bot}{\Gamma \vdash A} \qquad \wedge\textsc{-Intro}\ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \wedge\textsc{-Elim-1}\ \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \wedge\textsc{-Elim-2}\ \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \qquad \Rightarrow\textsc{-Intro}\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

$$\Rightarrow\textsc{-Elim}\ \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B} \qquad \vee\textsc{-Intro-1}\ \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \vee\textsc{-Intro-2}\ \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \qquad \vee\textsc{-Elim}\ \frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

Fig. 1. Sequent-style natural deduction for IPL

In the cover semantics of IPL, truth of formulas is defined using a gadget called a *cover system*. A cover system $C = (W, \sqsubseteq, \lhd)$ is a tuple consisting of a set W of worlds, a reflexive-transitive *refinement* relation $\sqsubseteq$ on W, and a *covering* relation $\lhd \subseteq W \times \mathcal{P}(W)$ subject to certain conditions. We write $w \sqsubseteq w'$ or $w' \sqsupseteq w$, saying w' *refines* w, to denote that the relation $\sqsubseteq$ relates the world w to the world w'. Similarly, we write $w \lhd \alpha$ or $\alpha \rhd w$, saying w is *covered by* α or α *covers* w or α is a *cover of* w, to denote that the covering relation $\lhd$ relates the world w to a set α consisting of worlds.

We can define a refinement relation $\preceq\ \subseteq \mathcal{P}(W) \times \mathcal{P}(W)$ on subsets of worlds using the refinement relation $\sqsubseteq$ on worlds as: $\alpha \preceq \alpha'$ if and only if (iff) for all worlds $v' \in \alpha'$ there exists a world $v \in \alpha$ such that $v \sqsubseteq v'$. We write $\alpha \preceq \alpha'$ or $\alpha' \succeq \alpha$, while saying α' *refines* α. The conditions on a cover system are:

- *Refinability*: If $w' \sqsupseteq w \lhd \alpha$, then there exists an α' such that $w' \lhd \alpha' \succeq \alpha$.
- *Inclusion*: If $w \lhd \alpha$, then $\{w\} \preceq \alpha$
- *Identity*: $w \lhd \{w\}$
- *Transitivity*: If $w \lhd \alpha$ and for all $v \in \alpha$ there exists an α_v such that $v \lhd \alpha_v$, then $w \lhd \bigcup_{v \in \alpha} \alpha_v$

We may intuitively understand a world w as a "state of knowledge", the refinement $w \sqsubseteq w'$ as increase in knowledge from w to w', and a cover α of a world w as defining a "locality" of knowledge states capturing knowledge local to w. Under this reading, the refinability condition ensures that local knowledge improves with increase in "current" knowledge: if $w \sqsubseteq w'$ and $w \lhd \alpha$, then some cover of w' refines α. Similarly, the inclusion condition states that knowledge local to a world w must refine current knowledge at w.

A *cover model* $\mathcal{M} = (C, V)$ for IPL couples a cover system C with a valuation function V mapping propositional atoms to *localized up-sets* of W, i.e. a function $V : \textsf{Atom} \to \mathcal{P}(W)$ satisfying the conditions:

- *Upper set*: if $w \sqsubseteq w'$ and $w \in V(p)$, then $w' \in V(p)$
- *Localization*: if $\exists \alpha.\, w \lhd \alpha \subseteq V(p)$, then $w \in V(p)$

The valuation function V maps a propositional atom p to a subset of worlds $V(p) \subseteq W$ where p is true. The conditions respectively state that $V(p)$ must be an up-set of the preorder $(W, \sqsubseteq)$, and that if p is "locally true" at w, i.e. at all worlds in some cover α of w, then it must be true at w.

Given a cover model $\mathcal{M}$, the truth of a formula A is given using the *satisfaction* relation $\Vdash$, for an arbitrary world $w \in W$ underlying the model $\mathcal{M}$, by induction on the formula as follows:

$$\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & w \in V(p) \\
\mathcal{M}, w \Vdash \top & \text{iff} & \text{true} \\
\mathcal{M}, w \Vdash \bot & \text{iff} & w \lhd \emptyset \\
\mathcal{M}, w \Vdash A \wedge B & \text{iff} & \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B \\
\mathcal{M}, w \Vdash A \vee B & \text{iff} & \exists \alpha.\, w \lhd \alpha \text{ and } \forall v \in \alpha.\, \mathcal{M}, v \Vdash A \text{ or } \mathcal{M}, v \Vdash B \\
\mathcal{M}, w \Vdash A \Rightarrow B & \text{iff} & \forall w' \sqsupseteq w.\, \mathcal{M}, w' \Vdash A \text{ implies } \mathcal{M}, w' \Vdash B
\end{array}$$

We extend the satisfaction relation to contexts and write $\mathcal{M}, w \Vdash \Gamma$ to denote $\mathcal{M}, w \Vdash A_i$ for all formulas A_i with $1 \leq i \leq n$ in $\Gamma = A_1, \ldots, A_n$. We define the *truth set* $|A|^{\mathcal{M}}$ of a formula A in some model $\mathcal{M}$ as the subset of worlds where A is true, and likewise extend this definition to contexts as follows:

$$|A|^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \Vdash A\} \qquad |A_1, A_2, \ldots, A_n|^{\mathcal{M}} = |A_1|^{\mathcal{M}} \cap |A_2|^{\mathcal{M}} \ldots \cap |A_n|^{\mathcal{M}}$$

We sometimes omit the subscript $\mathcal{M}$ and write $|A|$ or $|\Gamma|$ when it is evident from the context which model we are working with. We write $\Gamma \models_{\mathcal{M}} A$, saying Γ *entails* A *in model* $\mathcal{M}$, to denote that $|\Gamma|^{\mathcal{M}} \subseteq |A|^{\mathcal{M}}$. In other words, $\Gamma \models_{\mathcal{M}} A$ if and only if $\mathcal{M}, w \Vdash \Gamma$ implies $\mathcal{M}, w \Vdash A$ for all worlds w in model $\mathcal{M}$. Furthermore, we write $\Gamma \models A$, saying Γ *entails* A, to denote $\Gamma \models_{\mathcal{M}} A$ for all models $\mathcal{M}$.

To prove soundness for IPL, we begin with the following lemma, which observes that the conditions imposed on the truth of atoms are also satisfied by truth sets of arbitrary formulas and contexts.

Lemma 2.1 *For any formula A and cover model $\mathcal{M}$ of IPL, the truth set $|A|^{\mathcal{M}}$ is a localized up-set. In other words, it satisfies the following properties. It follows that $|\Gamma|^{\mathcal{M}}$ is a localized up-set for any context Γ.*

- *Upper set: if $w \sqsubseteq w'$ and $w \in |A|^{\mathcal{M}}$, then $w' \in |A|^{\mathcal{M}}$*
- *Localization: if $\exists \alpha.\, w \lhd \alpha \subseteq |A|^{\mathcal{M}}$, then $w \in |A|^{\mathcal{M}}$*

Proof. By induction on formula A and context Γ. □

Proposition 2.2 (Soundness for IPL) *If $\Gamma \vdash A$ holds, then so does $\Gamma \models A$.*

Proof. By induction on the derivation of $\Gamma \vdash A$, using Lemma 2.1 where needed (see Appendix A). □

We now turn our attention to proving completeness. Following the standard practice, we will achieve this by constructing a *canonical* model $\mathcal{N}$ that equates entailment in the model $\Gamma \models_{\mathcal{N}} A$ with derivability of the corresponding judgment $\Gamma \vdash A$. We begin with a few definitions for this purpose.

Definition 2.3 Given a formula A, we define the set $\mathit{Ant}\,(A)$, called the *antecedents* of A, as the set of contexts that A can be proved under the assumption of, i.e. $\mathit{Ant}\,(A) = \{\Gamma \in \mathit{Ctx} \mid \Gamma \vdash A\}$.

Definition 2.4 Define a cover system $C_{\text{IPL}} = (\mathit{Ctx}, \subseteq, \lhd_{\text{IPL}})$ by taking the set Ctx of contexts for worlds W, the context inclusion relation $\subseteq$ for the partial order relation $\sqsubseteq$, and the below inductively defined relation $\lhd_{\text{IPL}} \subseteq \mathit{Ctx} \times \mathcal{P}(\mathit{Ctx})$ for the covering relation $\lhd$. The relation $\lhd_{\text{IPL}}$ can be verified to satisfy the refinability, inclusion, identity and transitivity conditions by induction on its definition.

$$\Gamma \lhd_{\text{IPL}} \{\Gamma\} \qquad \frac{\Gamma \in \mathit{Ant}\,(\bot)}{\Gamma \lhd_{\text{IPL}} \emptyset} \qquad \frac{\Gamma \in \mathit{Ant}\,(A \vee B) \qquad \Gamma, A \lhd_{\text{IPL}} \alpha_1 \qquad \Gamma, B \lhd_{\text{IPL}} \alpha_2}{\Gamma \lhd_{\text{IPL}} \alpha_1 \cup \alpha_2}$$

Lemma 2.5 (Truth Lemma) *The tuple $\mathcal{N} = (C_{IPL}, \mathit{Ant})$ is a cover model for IPL with the characteristic property that, for every formula A and context Γ, we have $\Gamma \in |A|^{\mathcal{N}}$ if and only if $\Gamma \in \mathit{Ant}\,(A)$.*

Proof. We first check that the valuation $\mathit{Ant}\,(p)$ of an arbitrary atom p is a localized up-set by induction on the covering relation $\lhd_{\mathrm{IPL}}$, and then show the "characteristic" property by induction on the formula A.□

Theorem 2.6 (Completeness for IPL) *If* $\Gamma \models A$*, then* $\Gamma \vdash A$.

Proof. We first show $\Gamma \in \mathit{Ant}\,(\bigwedge \Gamma)$, equivalently $\Gamma \vdash \bigwedge \Gamma$, using the inference rules for IPL by induction on the context Γ. By applying Lemma 2.5 (from right to left of the bi-implication) we infer $\Gamma \in |\bigwedge \Gamma|^{\mathcal{N}}$. Moreover, we observe $|\bigwedge \Gamma|^{\mathcal{N}} = |\Gamma|^{\mathcal{N}}$ by appealing to the definition of truth sets and further infer $\Gamma \in |\Gamma|^{\mathcal{N}}$.

Given $\Gamma \models A$, we have $|\Gamma|^{\mathcal{N}} \subseteq |A|^{\mathcal{N}}$ since $\mathcal{N}$ is a model of IPL. This means we have $\Gamma \in |A|^{\mathcal{N}}$, and by applying Lemma 2.5 once again (from left to right), we conclude $\Gamma \in \mathit{Ant}\,(A)$ and thus $\Gamma \vdash A$. □

2.2 Relational Cover Models for IMLs

The language of the intuitionistic modal logic CM extends that of IPL with a unary connective $\heartsuit$, and its natural deduction proof system extends that of IPL with a rule CM/$\heartsuit$-Mon (for "monotonicity").

$$\mathit{Prop}\quad A, B := \ldots \mid \heartsuit A \qquad \frac{\Gamma \vdash \heartsuit A \qquad A \vdash B}{\Gamma \vdash \heartsuit B}\ \text{CM/}\heartsuit\text{-Mon}$$

A *relational cover system* (C, R) extends the definition of a cover system $C = (W, \sqsubseteq, \lhd)$ with an *accessibility* relation R. The relation R is a binary relation on worlds subject to the modal refinability and localization conditions stated below. We write $w\ R\ v$ or $v\ R^{-1}\ w$, and say w *can access* v or v is *accessible from* w, to denote that the world w is related to world v via the relation R. To state the modal conditions, we define an operator $\langle R \rangle$ on subsets of W, for a given $X \subseteq W$, as:

$$\langle R \rangle X = \{w \in W \mid \exists x \in X.\, w\ R\ x\}$$

The set $\langle R \rangle X$ identifies all worlds that can access some world in X. The modal conditions are:

- *Modal Refinability*: If $w' \sqsupseteq w\ R\ v$, then there exists a v' such that $w'\ R\ v' \sqsupseteq v$
- *Modal Localization*: If $w \lhd \alpha \subseteq \langle R \rangle X$, then there exists a v and α_v such that $w\ R\ v \lhd \alpha_v \subseteq X$.

A *relational cover model* $\mathcal{M} = (C, R, V)$ for CM couples a relational cover system (C, R) with a valuation function V mapping propositional atoms to localized up-sets of W—as before with IPL. The truth of CM formulas is defined by extending the satisfaction relation for IPL to modal formulas as follows:

$$\mathcal{M}, w \Vdash \heartsuit A \text{ iff } \exists v.\, w\ R\ v \text{ and } \mathcal{M}, v \Vdash A$$

To prove soundness for CM, we re-establish Lemma 2.1 by showing that truth sets for formulas in CM are indeed localized up-sets, using the modal conditions for the case of modal formulas $\heartsuit A$. We then prove soundness for CM by induction on the derivations of judgments, as in the proof of Proposition 2.2. The trouble, however, lies in proving completeness. Constructing a canonical relational cover model for CM requires us to extend the cover system in Definition 2.4 with a relation on contexts. A natural candidate for a relation in the canonical model $\mathcal{N}$ would be the relation $R_{\mathrm{CM}} \subseteq \mathit{Ctx} \times \mathit{Ctx}$ defined as follows:

$$\Gamma\ R_{\mathrm{CM}}\ \Delta \text{ iff there exists a formula } A \text{ s.t. } \Gamma \in \mathit{Ant}\,(\heartsuit A) \text{ and } \Delta = \{A\}$$

The relation R_{CM} has the essential character of equating entailment in the model with provability. We can show that the truth set $|\heartsuit B|$ determined by the relation R_{CM} is in fact equivalent to the set $\mathit{Ant}\,(\heartsuit B)$. However, R_{CM} crucially fails to satisfy modal localization, blocking us from using it to construct a relational cover model—inhibiting a proof of the truth lemma (Lemma 2.5) used to show completeness. Goldblatt [28, Section 8] encounters a similar roadblock in an attempt to construct a Henkin-style model for PLL.

At first sight, it may appear as though the modal localization condition is at fault. However, the modal localization condition simply states what is needed to show that the truth sets of modal formulas satisfy the localization property. Based on observations from the model constructions to follow in this article, our experience suggests that the interpretation given by relational cover semantics is itself somewhat restrictive and inconvenient. An accessibility relation forces us to pick *exactly* one possible future world, while many models necessitate a collection, or a "neighborhood", of possible future worlds. In the upcoming sections, we develop a conservative extension of relational cover semantics by replacing the accessibility relation R with an additional *modal* covering relation $\blacktriangleleft$ which alleviates this restriction.

2.3 Two-Cover Models for IMLs

A *two-cover system* $(C, \blacktriangleleft)$ extends the definition of a cover system $C = (W, \sqsubseteq, \lhd)$ with a *modal covering relation* $\blacktriangleleft$ subject to the modal refinability and localization conditions stated below. We may intuitively understand a modal cover β of a world w, written $w \blacktriangleleft \beta$ or $\beta \blacktriangleright w$, as defining a "speculation" about possible "future" states of knowledge based on "current" knowledge at w. To state the modal conditions, we define two operators $\langle\lhd\rangle$ and $\langle\blacktriangleleft\rangle$ on subsets of W, for a given $X \subseteq W$, as:

$$\langle\lhd\rangle X = \{w \in W \mid \exists\alpha.\, w \lhd \alpha \subseteq X\} \qquad \langle\blacktriangleleft\rangle X = \{w \in W \mid \exists\alpha.\, w \blacktriangleleft \alpha \subseteq X\}$$

The set $\langle\lhd\rangle(X)$ identifies all worlds w locally covered by some subset α of X, and similarly the set $\langle\blacktriangleleft\rangle X$ identifies all worlds w that are modally covered by some subset β of X. The modal conditions are:

- *Modal Refinability*: If $w' \sqsupseteq w \blacktriangleleft \alpha$, then there exists an α' such that $w' \blacktriangleleft \alpha' \succeq \alpha$.
- *Modal Localization*: If $w \lhd \alpha \subseteq \langle\blacktriangleleft\rangle X$, then there exists a β such that $w \blacktriangleleft \beta \subseteq \langle\lhd\rangle X$

The modal conditions on two-cover systems generalize the modal conditions on relational cover systems by respectively replacing the accessibility relation R and operator $\langle R\rangle$ with covering relation $\blacktriangleleft$ and operator $\langle\blacktriangleleft\rangle$. As before, the modal conditions ensure that truth sets of modal formulas are localized up-sets.

A *two-cover model* $\mathcal{M} = (C, \blacktriangleleft, V)$ for CM couples a two-cover system $(C, \blacktriangleleft)$ with a valuation function V that maps atoms to localized up-sets of W, which is a function $V : \mathit{Atom} \to \mathcal{P}(W)$ satisfying the upper set and localization conditions imposed on a cover model of IPL. Observe that there is no modal counterpart to the localization condition on V concerning the local covering relation $\lhd$. Intuitively, this is because we cannot expect a formula A that is "speculatively true" at a world w to become true at w. The truth of CM formulas is defined for a given two-cover model $\mathcal{M}$ of CM by extending the definition of the satisfaction relation for IPL to modal formulas as follows:

$$\mathcal{M}, w \Vdash \heartsuit A \text{ iff } \exists\beta.\, w \blacktriangleleft \beta \text{ and } \forall v \in \beta.\, \mathcal{M}, v \Vdash A$$

This definition states that a modal formula $\heartsuit A$ is true a world w iff A is true at all members v of some cover β of w. In contrast, recollect that the relational approach in the previous subsection requires A to be true at exactly one world v accessible from w. This means we can recover the relational semantics for CM from the two-cover semantics for CM by simply restricting all covers to be singletons.

Lemma 2.7 *For any two-cover model $\mathcal{M}$ of CM, the truth sets $|A|^{\mathcal{M}}$ and $|\Gamma|^{\mathcal{M}}$ are localized up-sets.*

Proof. By repeating the induction in Lemma 2.1, using the modal conditions for modal formulas. □

Proposition 2.8 (Soundness for CM) *If $\Gamma \vdash A$, then $\Gamma \models A$.*

Proof. By repeating the induction in proof of Proposition 2.2, now using Lemma 2.7. The interesting case is that of *Rule CM/♡-Mon*: We must show $|\Gamma| \subseteq |\heartsuit B|$ from the induction hypotheses $|\Gamma| \subseteq |\heartsuit A|$ (IH.1) and $|A| \subseteq |B|$ (IH.2). If some $w \in |\Gamma|$, it follows from IH.1 that $w \in |\heartsuit A|$, which means for some β, $w \blacktriangleleft \beta$ and $\beta \subseteq |A|$. It follows from IH.2 that $\beta \subseteq |B|$, which means we also have $w \in |\heartsuit B|$ as desired. □

To prove completeness for CM, let us define a two-cover system $C_{\mathrm{CM}} = (C_{\mathrm{IPL}}, \blacktriangleleft_{\mathrm{CM}})$ coupling the cover system C_{IPL} (reproducing Definition 2.4 in the language of CM) with the modal covering relation $\blacktriangleleft_{\mathrm{CM}} \subseteq \mathit{Ctx} \times \mathcal{P}(\mathit{Ctx})$ defined inductively below. The relation $\blacktriangleleft_{\mathrm{CM}}$ can be verified to satisfy the modal refinability and localization conditions by induction on its definition.

$$\frac{\Gamma \in \mathit{Ant}\,(\heartsuit A)}{\Gamma \blacktriangleleft_{\mathrm{CM}} \{A\}} \qquad \frac{\Gamma \in \mathit{Ant}\,(\bot)}{\Gamma \blacktriangleleft_{\mathrm{CM}} \emptyset} \qquad \frac{\Gamma \in \mathit{Ant}\,(A \vee B) \quad \Gamma, A \blacktriangleleft_{\mathrm{CM}} \alpha_1 \quad \Gamma, B \blacktriangleleft_{\mathrm{CM}} \alpha_2}{\Gamma \blacktriangleleft_{\mathrm{CM}} \alpha_1 \cup \alpha_2}$$

Observe that there is an overlap in the definitions of the modal ($\blacktriangleleft_{\mathrm{CM}}$) and local ($\lhd_{\mathrm{IPL}}$) covering relations used to define C_{CM}. This overlap allows us to show that the modal localization condition holds for C_{CM}.

Lemma 2.9 (Truth Lemma) *The tuple $\mathcal{N} = (C_{CM}, \mathit{Ant})$ is a cover model for CM with the characteristic property that, for every formula A and context Γ, we have $\Gamma \in |A|^{\mathcal{N}}$ if and only if $\Gamma \in \mathit{Ant}(A)$.*

Proof. By repeating the induction on formulas in Lemma 2.5. The interesting case is that of modal formulas $\heartsuit B$. From left to right: if $\Gamma \in |\heartsuit B|$ then for some β, $\Gamma \blacktriangleleft_{\mathrm{CM}} \beta$ and $\beta \subseteq |B|$. By applying the induction hypothesis on B, we know $|B| \subseteq \mathit{Ant}(B)$, which means $\beta \subseteq \mathit{Ant}(B)$. By induction on the relation $\blacktriangleleft_{\mathrm{CM}}$, we can show that $\Gamma \blacktriangleleft_{\mathrm{CM}} \beta \subseteq \mathit{Ant}(B)$ implies $\Gamma \in \mathit{Ant}(\heartsuit B)$ as desired. From right to left: if $\Gamma \in \mathit{Ant}(\heartsuit B)$, then $\Gamma \blacktriangleleft_{\mathrm{CM}} \{B\}$. By applying the IH once again on B, we know $\mathit{Ant}(B) \subseteq |B|$, which means we have $\{B\} \subseteq |B|$ since $\{B\} \subseteq \mathit{Ant}(B)$. Altogether $\Gamma \blacktriangleleft_{\mathrm{CM}} \{B\} \subseteq |B|$ and thus $\Gamma \in |\heartsuit B|$. □

Theorem 2.10 (Completeness for CM) *If $\Gamma \models A$, then $\Gamma \vdash A$.*

Proof. By repeating the argument in Theorem 2.6, now using Lemma 2.9. □

3 Semantic Analysis of Two-Cover Models

The operators $\langle\lhd\rangle$ and $\langle\blacktriangleleft\rangle$ defined in the previous section possess a number of general algebraic properties that make it possible to avoid repetition in the proofs of soundness and completeness for various IMLs with respect to their two-cover semantics. We will identify these properties in this section.

Given a cover system $C = (W, \sqsubseteq, \lhd)$, recollect that an *upper set* or *up-set* is an "upwards closed" subset $X \subseteq W$ with the property that if $w \sqsubseteq w'$ and $w \in X$, then $w' \in X$. Up-sets can be characterized using an operator $\uparrow$ defined on subsets of W as $\uparrow X = \{w \in W \mid \exists x \in X.\, x \sqsubseteq w\}$. The set $\uparrow X$ identifies all worlds w that refine some world x in X, and X is an up-set if and only if $\uparrow X = X$. Recollect similarly that a *localized* or *localizing* set, is a subset $X \subseteq W$ with the property that if $\exists \alpha.\, w \lhd \alpha \subseteq X$, then $w \in X$. Localized sets can be characterized using the operator $\langle\lhd\rangle X = \{w \in W \mid \exists \alpha.\, w \lhd \alpha \subseteq X\}$ from earlier. The set $\langle\lhd\rangle X$ identifies all worlds w covered by some subset α of X, and a set X is a localized set if and only if $\langle\lhd\rangle X \subseteq X$. We refer to an up-set as a *localized up-set* if it also localized. We write $\mathcal{U}(W)$ to denote the collection of all up-sets and $\mathcal{LU}(W)$ to denote the collection of all localized up-sets.

Proposition 3.1 *The operator $\langle\lhd\rangle : \mathcal{P}(W) \to \mathcal{P}(W)$ exhibits the following properties:*

(i) *$\langle\lhd\rangle$ is a nucleus on subsets of W, i.e. it is monotone (preserves $\subseteq$) and for all $X, Y \in \mathcal{P}(W)$,*

$$X \cap \langle\lhd\rangle Y \subseteq \langle\lhd\rangle(X \cap Y) \qquad X \subseteq \langle\lhd\rangle X \qquad \langle\lhd\rangle\langle\lhd\rangle X \subseteq \langle\lhd\rangle X$$

(ii) *$\langle\lhd\rangle$ is a nucleus on up-sets of W, i.e. $\langle\lhd\rangle : \mathcal{U}(W) \to \mathcal{U}(W)$ is a nucleus*

(iii) *$\langle\lhd\rangle$ is a nucleus on localized up-sets W, i.e. $\langle\lhd\rangle : \mathcal{LU}(W) \to \mathcal{LU}(W)$ is a nucleus*

Proof. For property (i), monotonicity follows from definition of $\langle\lhd\rangle$, while the inequalities follow respectively from the reachability, identity and transitivity conditions. For property (ii), we use the refinability condition to show that $\langle\lhd\rangle X$ must be an up-set if X is. For property (iii), observe from (i) that $\langle\lhd\rangle\langle\lhd\rangle X \subseteq \langle\lhd\rangle X$ for any subset X, and thus $\langle\lhd\rangle X$ is localizing for a localized upset-set X. □

Recall that a Heyting algebra $H = (U, \leq, \times, +, 1, 0, \Rightarrow)$ is a lattice $(U, \leq, \times, +)$ consisting of a partial order $\leq$ on a carrier set U with meet ($\times$) and join ($+$) operations, accompanied by a maximal element 1, a minimal element 0, and an operation $\Rightarrow$ on U such that $c \leq a \Rightarrow b$ if and only if $c \times a \leq b$, for all elements $a, b, c \in U$. Further recall that an *algebraic model* $\mathcal{A} = (H, V)$ of IPL consists of a Heyting algebra H and valuation function $V : \mathit{Atom} \to U$ mapping atoms to elements of the set U. For any algebraic model $\mathcal{A}$ of IPL, we can extend the valuation of atoms to give an *interpretation* of formulas $[\![-]\!]^{\mathcal{A}} : \mathit{Prop} \to U$ and contexts $[\![-]\!]^{\mathcal{A}} : \mathit{Ctx} \to U$ in the carrier set U of the underlying Heyting algebra H as follows:

$$[\![p]\!]^{\mathcal{A}} = V(p) \qquad [\![A \wedge B]\!]^{\mathcal{A}} = [\![A]\!]^{\mathcal{A}} \times [\![B]\!]^{\mathcal{A}} \qquad [\![\cdot]\!]^{\mathcal{A}} = 1$$
$$[\![\top]\!]^{\mathcal{A}} = 1 \qquad [\![A \vee B]\!]^{\mathcal{A}} = [\![A]\!]^{\mathcal{A}} + [\![B]\!]^{\mathcal{A}} \qquad [\![\Gamma, A]\!]^{\mathcal{A}} = [\![\Gamma]\!]^{\mathcal{A}} \times [\![A]\!]^{\mathcal{A}}$$
$$[\![\bot]\!]^{\mathcal{A}} = 0 \qquad [\![A \Rightarrow B]\!]^{\mathcal{A}} = [\![A]\!]^{\mathcal{A}} \Rightarrow [\![B]\!]^{\mathcal{A}}$$

Moreover, it is well known that these functions are both sound and complete, meaning a judgment $\Gamma \vdash A$ is derivable in a proof system for IPL, if and only if, $[\![\Gamma]\!]^{\mathcal{A}} \leq [\![A]\!]^{\mathcal{A}}$ holds for all algebraic models $\mathcal{A}$ of IPL.

Proposition 3.2 *Every cover system $C = (W, \sqsubseteq, \lhd)$ determines a Heyting algebra $\widehat{C}$ defined by taking:*

- *$\mathcal{LU}(W)$ as the carrier ordered by set inclusion $\subseteq$*
- *$X \cap Y$ as the meet of localized up-sets X and Y*
- *$\langle\lhd\rangle(X \cup Y)$ as the join of localized up-sets X and Y*
- *W as the maximal element and $\langle\lhd\rangle(\emptyset)$ as the minimal element*
- *$X \Rightarrow Y = \{w \mid \uparrow\{w\} \cap X \subseteq Y\}$ as the exponent of localized up-sets X and Y*

Proof. Using the relevant definitions and properties of the operator $\langle\lhd\rangle$ in Proposition 3.1. □

Proposition 3.3 *Every cover model $\mathcal{M} = (C, V)$ of IPL determines an equivalent algebraic model $\widehat{\mathcal{M}} = (\widehat{C}, V)$ of IPL such that $|A|^{\mathcal{M}} = [\![A]\!]^{\widehat{\mathcal{M}}}$ and $|\Gamma|^{\mathcal{M}} = [\![\Gamma]\!]^{\widehat{\mathcal{M}}}$ for all formulas A and contexts Γ in IPL.*

Proof. It follows from Proposition 3.2 and the conditions on the function V (in a cover model of IPL) that $\widehat{\mathcal{M}}$ is indeed an algebraic model of IPL. This means we obtain an interpretation of a formula A as a localized up-set $[\![A]\!]^{\widehat{\mathcal{M}}}$. This interpretation can be given explicitly by induction on A, and extended to a context Γ, as follows:

$$
\begin{aligned}
[\![p]\!]^{\widehat{\mathcal{M}}} &= V(p) & [\![A \wedge B]\!]^{\widehat{\mathcal{M}}} &= [\![A]\!]^{\widehat{\mathcal{M}}} \cap [\![B]\!]^{\widehat{\mathcal{M}}} & [\![\cdot]\!]^{\widehat{\mathcal{M}}} &= W \\
[\![\top]\!]^{\widehat{\mathcal{M}}} &= W & [\![A \vee B]\!]^{\widehat{\mathcal{M}}} &= \langle\lhd\rangle([\![A]\!]^{\widehat{\mathcal{M}}} \cup [\![B]\!]^{\widehat{\mathcal{M}}}) & [\![\Gamma, A]\!]^{\widehat{\mathcal{M}}} &= [\![\Gamma]\!]^{\widehat{\mathcal{M}}} \cap [\![A]\!]^{\widehat{\mathcal{M}}} \\
[\![\bot]\!]^{\widehat{\mathcal{M}}} &= \langle\lhd\rangle(\emptyset) & [\![A \Rightarrow B]\!]^{\widehat{\mathcal{M}}} &= [\![A]\!]^{\widehat{\mathcal{M}}} \Rightarrow [\![B]\!]^{\widehat{\mathcal{M}}}
\end{aligned}
$$

It can readily observed by induction that for any A and Γ in IPL, $|A|^{\mathcal{M}} = [\![A]\!]^{\widehat{\mathcal{M}}}$ and $|\Gamma|^{\mathcal{M}} = [\![\Gamma]\!]^{\widehat{\mathcal{M}}}$. □

We now turn our attention to the *modal* operator $\langle\blacktriangleleft\rangle$. Given a two-cover system $(C, \blacktriangleleft)$, recollect that the operator $\langle\blacktriangleleft\rangle$ is defined on subsets of W as $\langle\blacktriangleleft\rangle X = \{w \in W \mid \exists\alpha.\, w \blacktriangleleft \alpha \subseteq X\}$.

Proposition 3.4 *The operator $\langle\blacktriangleleft\rangle$ is a monotone function $\langle\blacktriangleleft\rangle : \mathcal{LU}(W) \to \mathcal{LU}(W)$ on localized up-sets*

Proof. While monotonicity holds readily, we must show $\langle\blacktriangleleft\rangle X$ is a localized up-set whenever X is.

To show $\langle\blacktriangleleft\rangle X$ is an up-set, suppose $w \sqsubseteq w'$ and $w \in \langle\blacktriangleleft\rangle X$. This means for some β, $w' \sqsupseteq w \blacktriangleleft \beta \subseteq X$. Due to the modal refinability condition, we know that for some β', we have $w' \blacktriangleleft \beta' \succeq \beta$. Since X is an up-set and β' refines β, we also have $\beta' \subseteq X$, and thus $w' \blacktriangleleft \beta' \subseteq X$, which is why $w' \in \langle\blacktriangleleft\rangle X$.

To show $\langle\blacktriangleleft\rangle X$ is localizing, recollect that the modal localization effectively states $\langle\lhd\rangle\langle\blacktriangleleft\rangle X \subseteq \langle\blacktriangleleft\rangle\langle\lhd\rangle X$. Since X is localizing, we know $\langle\lhd\rangle X \subseteq X$, which implies $\langle\lhd\rangle\langle\blacktriangleleft\rangle X \subseteq \langle\blacktriangleleft\rangle X$ since $\langle\blacktriangleleft\rangle$ is monotonic. □

A *modal Heyting algebra* (H, m) is a Heyting algebra H accompanied by a monotone function $m : U \to U$ on the carrier set U underlying H. An algebraic model $\mathcal{A} = (H, m, V)$ of CM consists of a modal Heyting algebra (H, m) and a valuation function $V : \mathsf{Atom} \to U$. As before, it can be shown that interpretation of formulas $[\![-]\!]^{\mathcal{A}} : \mathsf{Prop} \to U$ and contexts $[\![-]\!]^{\mathcal{A}} : \mathsf{Ctx} \to U$ in CM are sound and complete for the judgments $\Gamma \vdash A$ in CM by taking $[\![\heartsuit A]\!]^{\mathcal{A}} = m([\![A]\!]^{\mathcal{A}})$ for the case of modal formulas $\heartsuit A$ in CM.

Proposition 3.5 *Every two-cover model $\mathcal{M} = (C, \blacktriangleleft, V)$ of CM determines an equivalent algebraic model $\widehat{\mathcal{M}} = (\widehat{C}, \langle\blacktriangleleft\rangle, V)$ of CM, with $\widehat{C}$ as in Proposition 3.3, s.t. $|A|^{\mathcal{M}} = [\![A]\!]^{\widehat{\mathcal{M}}}$ for any formula A in CM.*

Proof. Follows from Propositions 3.3 and 3.4 and the observation $[\![\heartsuit A]\!]^{\widehat{\mathcal{M}}} = \langle\blacktriangleleft\rangle[\![A]\!]^{\widehat{\mathcal{M}}}$. □

4 Two-Cover Semantics for IMLs

4.1 Minimal Lax Logic

The language of SL extends that of IPL with a unary connective $\lozenge$ known as the lax modality. A modal formula $\lozenge A$ may be intuitively understood as denoting the truth of formula A qualified by some constraint, i.e. "possibly A". The logic SL admits the characteristic axiom $\mathrm{S} : A \wedge \Diamond B \Rightarrow \Diamond(A \wedge B)$, which states that if A is true and B is possibly true, then both A and B are possibly true. The proof rules for SL extend those of IPL with a rule SL/$\Diamond$-Map (for "mapping").

$$\mathit{Prop}\quad A, B ::= \ldots \mid \Diamond A \qquad \frac{\Gamma \vdash \Diamond A \qquad \Gamma, A \vdash B}{\Gamma \vdash \Diamond B}\ \text{SL}/\Diamond\text{-Map}$$

An *SL algebra* is a modal Heyting algebra (H, m) where the monotone function $m : U \to U$ satisfies the inequality $a \times m(b) \leq m(a \times b)$, for all $a, b \in U$. We may equivalently characterize an SL algebra "equationally" (as in [4, Definition 4]) by dropping the monotonicity condition on the function m in favor of an additional inequality $m(a) \leq m(a + b)$. An algebraic model $\mathcal{A} = (H, m, V)$ of SL consists of an SL algebra (H, m) and a valuation function $V : \mathit{Atom} \to U$ mapping atoms to the carrier set U underlying H. The interpretation of formulas in SL can be given by extending the interpretation $[\![-]\!] : \mathit{Prop} \to U$ of formulas in IPL with $[\![\Diamond A]\!]^{\mathcal{A}} = m([\![A]\!]^{\mathcal{A}})$ for the case of modal formulas $\Diamond A$ in SL.

It can further be shown by induction that if a judgment $\Gamma \vdash A$ is derivable in SL, then $[\![\Gamma]\!]^{\mathcal{A}} \leq [\![A]\!]^{\mathcal{A}}$ holds for all algebraic models $\mathcal{A}$ of SL. The interesting case is that of Rule SL/$\Diamond$-Map. By applying the induction hypothesis to the premises of the rule, we obtain the inequalities $[\![\Gamma]\!] \leq m[\![A]\!]$ (IH.1) and $[\![\Gamma]\!] \times [\![A]\!] \leq [\![B]\!]$ (IH.2). It follows from IH.1 that $[\![\Gamma]\!] \leq ([\![\Gamma]\!] \times m[\![A]\!]) \leq m([\![\Gamma]\!] \times [\![A]\!])$ for an SL algebra, which when combined with IH.2 and monotonicity of m gives us the inequality $[\![\Gamma]\!] \leq m[\![B]\!]$ as desired.

A two-cover model $\mathcal{M} = (C, \blacktriangleleft, V)$ of SL consists of a two-cover system $(C, \blacktriangleleft)$ and a valuation function $V : \mathit{Atom} \to \mathcal{LU}(W)$, where the modal covering relation $\blacktriangleleft$ satisfies, in addition to the usual modal refinability and localization conditions, a *modal inclusion* condition stated below:

- *Modal Inclusion*: If $w \blacktriangleleft \alpha$, then $\{w\} \preceq \alpha$

The truth of modal formulas for an arbitrary two-cover model $\mathcal{M}$ of SL is given as before for CM by extending the satisfaction relation to modal formulas $\Diamond A$ in a manner that ensures $|\Diamond A|^{\mathcal{M}} = \langle\blacktriangleleft\rangle|A|^{\mathcal{M}}$.

$$\mathcal{M}, w \Vdash \Diamond A \text{ iff } \exists \beta.\, w \blacktriangleleft \beta \text{ and } \forall v \in \beta.\, \mathcal{M}, v \Vdash A$$

Proposition 4.1 *Every two-cover model $\mathcal{M} = (C, \blacktriangleleft, V)$ of SL determines an equivalent algebraic model $\widehat{\mathcal{M}} = (\widehat{C}, \langle\blacktriangleleft\rangle, V)$ of SL, whose underlying Heyting algebra $\widehat{C}$ is given by localized up-sets as in Proposition 3.3, such that $|A|^{\mathcal{M}} = [\![A]\!]^{\widehat{\mathcal{M}}}$ and $|\Gamma|^{\mathcal{M}} = [\![\Gamma]\!]^{\widehat{\mathcal{M}}}$ for all formulas A and contexts Γ in SL.*

Proof. Every two-cover system $(C, \blacktriangleleft)$ determines a modal Heyting algebra $(\widehat{C}, \langle\blacktriangleleft\rangle)$, as in the proof of Proposition 3.5 due to Propositions 3.3 and 3.4. To show $(\widehat{C}, \langle\blacktriangleleft\rangle)$ is also an SL algebra, it remains to show $X \cap \langle\blacktriangleleft\rangle Y \subseteq \langle\blacktriangleleft\rangle(X \cap Y)$ for all $X, Y \in \mathcal{LU}(W)$, which we achieve using the modal inclusion condition.

Observe that the interpretation of formulas in $\widehat{\mathcal{M}}$ readily satisfies the equality $[\![\Diamond A]\!]^{\widehat{\mathcal{M}}} = \langle\blacktriangleleft\rangle[\![A]\!]^{\widehat{\mathcal{M}}}$ by definition. As a result, we can once again show that $[\![A]\!]^{\widehat{\mathcal{M}}} = |A|^{\mathcal{M}}$ and $[\![\Gamma]\!]^{\widehat{\mathcal{M}}} = |\Gamma|^{\mathcal{M}}$ by induction on A and Γ. For the case of modal formulas $\Diamond A$, we observe that $[\![\Diamond A]\!]^{\widehat{\mathcal{M}}} = \langle\blacktriangleleft\rangle[\![A]\!]^{\widehat{\mathcal{M}}} = \langle\blacktriangleleft\rangle|A|^{\mathcal{M}} = |\Diamond A|^{\mathcal{M}}$. □

Proposition 4.2 (Soundness for SL) *If $\Gamma \vdash A$, then $\Gamma \models A$.*

Proof. By soundness of SL for its algebraic models and Proposition 4.1, we have $\Gamma \vdash A$ implies $[\![\Gamma]\!]^{\mathcal{M}} \subseteq [\![A]\!]^{\mathcal{M}}$ for all two-cover models $\mathcal{M}$. Since the algebraic interpretation of formulas and contexts in SL is equivalent to their respective truth sets, we also have $|\Gamma|^{\mathcal{M}} \subseteq |A|^{\mathcal{M}}$, and thus $\Gamma \models A$. □

As before with CM, to prove completeness for SL we construct a canonical two-cover model $\mathcal{N}$ that equates entailment of formulas in the model $\mathcal{N}$ to provability in SL. For this purpose, let us define a two-cover system $C_{\mathrm{SL}} = (C_{\mathrm{IPL}}, \blacktriangleleft_{\mathrm{SL}})$ coupling the cover system C_{IPL} (reproducing Definition 2.4 in the language of SL) with the modal covering relation $\blacktriangleleft_{\mathrm{SL}} \subseteq \mathit{Ctx} \times \mathcal{P}(\mathit{Ctx})$ defined inductively below.

$$\frac{\Gamma \in \mathit{Ant}\,(\Diamond A)}{\Gamma \blacktriangleleft_{\mathrm{SL}} \{\Gamma, A\}} \qquad \frac{\Gamma \in \mathit{Ant}\,(\bot)}{\Gamma \blacktriangleleft_{\mathrm{SL}} \emptyset} \qquad \frac{\Gamma \in \mathit{Ant}\,(A \vee B) \qquad \Gamma, A \blacktriangleleft_{\mathrm{SL}} \alpha_1 \qquad \Gamma, B \blacktriangleleft_{\mathrm{SL}} \alpha_2}{\Gamma \blacktriangleleft_{\mathrm{SL}} \alpha_1 \cup \alpha_2}$$

As before with the relation $\blacktriangleleft_{\mathrm{CM}}$, the relation $\blacktriangleleft_{\mathrm{SL}}$ can be shown to satisfy the modal refinability and localization conditions. In contrast to $\blacktriangleleft_{\mathrm{CM}}$, however, $\blacktriangleleft_{\mathrm{SL}}$ crucially also satisfies the modal inclusion condition. This is because all contexts Γ' in a cover $\alpha \blacktriangleright_{\mathrm{SL}} \Gamma$ subsume Γ, i.e. $\Gamma \subseteq \Gamma'$, and thus $\{\Gamma\} \preceq \alpha$.

Lemma 4.3 (Truth Lemma) *The tuple $\mathcal{N} = (C_{SL}, \mathsf{Ant})$ is a cover model of SL s.t. $|A|^{\mathcal{N}} = \mathsf{Ant}\,(A)$.*

Theorem 4.4 (Completeness for SL) *If $\Gamma \models A$, then $\Gamma \vdash A$.*

Proof. By repeating the argument in Theorem 2.10, now using Lemma 4.3. □

4.2 Propositional Lax Logic

The logic PLL extends the axioms of SL with the axioms R : $A \Rightarrow \Diamond A$ and J : $\Diamond\Diamond A \Rightarrow \Diamond A$. The language of PLL extends that of IPL with a unary connective $\Diamond$, as with the language of SL, while the proof system for PLL extends that of IPL with the rules PLL/$\Diamond$-Intro and PLL/$\Diamond$-Bind (for "binding") given below.

$$\mathsf{Prop}\quad A, B := \ldots \mid \Diamond A \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\ \textsc{PLL}/\Diamond\text{-Intro} \qquad \frac{\Gamma \vdash \Diamond A \qquad \Gamma, A \vdash \Diamond B}{\Gamma \vdash \Diamond B}\ \textsc{PLL}/\Diamond\text{-Bind}$$

A *PLL algebra* is an SL algebra (H, m) where the monotone function $m : U \to U$ is inflationary and idempotent, i.e. it additionally satisfies the inequalities $a \leq m(a)$ and $m(m(a)) \leq m(a)$, for all $a \in U$. We may equivalently characterize a PLL algebra without reference to SL algebras as a Heyting algebra H accompanied by a nucleus operator m. An algebraic model $\mathcal{A} = (H, m, V)$ of PLL consists of a PLL algebra (H, m) and a valuation function $V : \mathsf{Atom} \to U$, where the interpretation of modal formulas is given by $[\![\Diamond A]\!]^{\mathcal{A}} = m([\![A]\!]^{\mathcal{M}})$. It is known moreover that $\Gamma \vdash A$ in PLL if and only if $[\![\Gamma]\!]^{\mathcal{A}} \leq [\![A]\!]^{\mathcal{A}}$ for all algebraic models $\mathcal{M}$ of PLL [4,28].

A two-cover model $\mathcal{M} = (C, \blacktriangleleft, V)$ of PLL is a two-cover model of SL that additionally satisfies the modal *identity* and *transitivity* conditions stated below:

- *Modal Identity*: $w \blacktriangleleft \{w\}$
- *Modal Transitivity*: If $w \blacktriangleleft \alpha$ and for all $v \in \alpha$ there exists an α_v such that $v \blacktriangleleft \alpha_v$, then $w \blacktriangleleft \bigcup_{v \in \alpha} \alpha_v$

The truth of modal formulas in a two-cover model $\mathcal{M}$ of PLL is identical to SL, i.e. $|\Diamond A|^{\mathcal{M}} = \langle\blacktriangleleft\rangle |A|^{\mathcal{M}}$.

As before with SL, every two-cover model $\mathcal{M} = (C, \blacktriangleleft, V)$ of PLL determines an equivalent algebraic model $\widehat{\mathcal{M}} = (\widehat{C}, \langle\blacktriangleleft\rangle, V)$ of PLL, i.e. $|A|^{\mathcal{M}} = [\![A]\!]^{\widehat{\mathcal{M}}}$ and $|\Gamma|^{\mathcal{M}} = [\![\Gamma]\!]^{\widehat{\mathcal{M}}}$ for all formulas A and contexts Γ in PLL. The operator $\langle\blacktriangleleft\rangle$ is a nucleus on $\mathcal{LU}(W)$—the carrier set of the Heyting algebra $\widehat{C}$—since its underlying modal covering relation $\blacktriangleleft$ satisfies the conditions (refinability, inclusion, identity and transitivity) imposed on the local covering relation $\lhd$ underlying the nucleus operator $\langle\lhd\rangle$ (c.f. Proposition 3.1).

Proposition 4.5 (Soundness for PLL) *If $\Gamma \vdash A$, then $\Gamma \models A$.*

Proof. Every two-cover model of PLL determines an equivalent algebraic model of PLL, which we know is sound for PLL (repeating the argument in Proposition 4.2). Thus two-cover semantics is as well sound. □

To prove completeness for PLL, we define a two-cover system $C_{\mathrm{PLL}} = (C_{\mathrm{IPL}}, \blacktriangleleft_{\mathrm{PLL}})$ coupling the cover system C_{IPL} from earlier with the modal covering relation $\blacktriangleleft_{\mathrm{PLL}} \subseteq \mathsf{Ctx} \times \mathcal{P}(\mathsf{Ctx})$ defined below.

$$\Gamma \blacktriangleleft_{\mathrm{PLL}} \{\Gamma\} \qquad \frac{\Gamma \in \mathsf{Ant}\,(\Diamond A) \qquad \Gamma, A \blacktriangleleft_{\mathrm{PLL}} \alpha}{\Gamma \blacktriangleleft_{\mathrm{PLL}} \alpha}$$

$$\frac{\Gamma \in \mathsf{Ant}\,(\bot)}{\Gamma \blacktriangleleft_{\mathrm{PLL}} \emptyset} \qquad \frac{\Gamma \in \mathsf{Ant}\,(A \vee B) \qquad \Gamma, A \blacktriangleleft_{\mathrm{PLL}} \alpha_1 \qquad \Gamma, B \blacktriangleleft_{\mathrm{PLL}} \alpha_2}{\Gamma \blacktriangleleft_{\mathrm{PLL}} \alpha_1 \cup \alpha_2}$$

The relation $\blacktriangleleft_{\mathrm{PLL}}$ readily satisfies the modal identity condition by definition, while modal transitivity and the remaining conditions can be shown by induction on its definition. The definition of $\blacktriangleleft_{\mathrm{PLL}}$ ensures that $\langle\triangleleft\rangle X \subseteq \langle\blacktriangleleft\rangle X$ for all subsets $X \subseteq \mathit{Ctx}$. The modal localization condition follows as a result, since $\langle\triangleleft\rangle\langle\blacktriangleleft\rangle X \subseteq \langle\blacktriangleleft\rangle\langle\blacktriangleleft\rangle X = \langle\blacktriangleleft\rangle X \subseteq \langle\blacktriangleleft\rangle\langle\triangleleft\rangle X$ since $\langle\blacktriangleleft\rangle$ and $\langle\triangleleft\rangle$ are also nuclei on $\mathcal{P}(\mathit{Ctx})$.

Theorem 4.6 (Completeness for PLL) *If* $\Gamma \models A$, *then* $\Gamma \vdash A$.

Proof. By showing the truth lemma for $\mathcal{N} = (C_{\text{PLL}}, \mathit{Ant})$ and repeating the argument in Theorem 4.4.□

4.3 Dual-context formulation of CK$_\Box$

The language of CK$_\Box$ extends that of IPL with a unary connective $\Box$ known as the box modality. A modal formula $\Box A$ may be read as "necessarily A" and intuitively understood as asserting that A is valid, i.e. universally true. The logic CK$_\Box$ admits the *necessitation* rule, which states that if A is valid then so is $\Box A$, and the characteristic axiom $\text{K} : \Box(A \Rightarrow B) \Rightarrow \Box A \Rightarrow \Box B$. A *dual-context* sequent-style proof system for CK$_\Box$, denoted DCK$_\Box$, is given using judgments $\Delta ; \Gamma \vdash A$ indexed by two contexts Δ and Γ. The "global" context Δ consists of formulas that are assumed to be valid, while the usual "local" context Γ consists of formulas that are assumed to be true for some specific world. The proof rules for the non-modal fragment can be given as before for IPL by leaving the global context untouched (see Appendix A). The proof rules for the modal fragment are given by the rules DCK$_\Box$/$\Box$-Intro and DCK$_\Box$/$\Box$-Elim defined below.

$$\mathit{Prop} \quad A, B ::= \ldots \mid \Box A \qquad \frac{\Delta ; \cdot \vdash A}{\Delta ; \Gamma \vdash \Box A}\ \text{DCK}_\Box/\Box\text{-Intro} \qquad \frac{\Delta ; \Gamma \vdash \Box A \qquad \Delta, A ; \Gamma \vdash B}{\Delta ; \Gamma \vdash B}\ \text{DCK}_\Box/\Box\text{-Elim}$$

A *CK*$_\Box$ *algebra* is a modal Heyting algebra (H, m) where the monotone function $m : U \to U$ preserves all finite meets, i.e. it satisfies the equations $m(1) = 1$ and $m(a \times b) = m(a) \times m(b)$, for all $a, b \in U$. An algebraic model $\mathcal{A} = (H, m, V)$ of DCK$_\Box$ consists of a CK$_\Box$ algebra (H, m) and a valuation function $V : \mathit{Atom} \to U$. The interpretation $[\![-]\!]^{\mathcal{A}}$ of formulas and contexts in the set U is defined as before for the previous logics, where the interpretation of modal formulas is given as $[\![\Box A]\!]^{\mathcal{A}} = m([\![A]\!]^{\mathcal{A}})$.

For some algebra $\mathcal{A}$ of CK$_\Box$, a formula A is said to be *algebraically valid* iff $[\![A]\!] = 1$ and *algebraically true* for an element (as opposed to world) $u \in U$ iff $u \leq [\![A]\!]$. Consequentially, a formula A is algebraically valid iff it is algebraically true for all elements. The requirement that m preserves all finite meets allows us to show that the necessitation rule and axiom K are algebraically sound principles. The equation $m(1) = 1$ allows us to show that the necessitation rule is algebraically sound: if a formula A is algebraically valid, meaning $[\![A]\!] = 1$, then $[\![\Box A]\!] = m([\![A]\!]) = m(1) = 1$, and thus $\Box A$ is also algebraically valid. Similarly, the equation $m(a) \times m(b) = m(a \times b)$ allows us to show that axiom $\text{K} : \Box(A \Rightarrow B) \Rightarrow \Box A \Rightarrow \Box B$ is algebraically valid, since it implies that the inequality $m(a \Rightarrow b) \times m(a) \leq m(b)$ holds for all elements $a, b \in U$.

If a dual-context judgment $\Delta ; \Gamma \vdash A$ is derivable in DCK$_\Box$, then the inequality $m([\![\Delta]\!]^{\mathcal{A}}) \times [\![\Gamma]\!]^{\mathcal{A}} \leq [\![A]\!]^{\mathcal{A}}$ must hold for all algebraic models $\mathcal{A}$ of DCK$_\Box$. This can be observed readily by induction on the derivation of judgment by using the equality $1 = m(1)$ for the case of Rule DCK$_\Box$/$\Box$-Intro and $m(a) \times m(b) = m(a \times b)$ for the case of Rule DCK$_\Box$/$\Box$-Elim. Alternatively, we may also appeal to the soundness of the categorical interpretation of DCK$_\Box$ [32, Section 6.2], where the modality $\Box$ is interpreted as an endofunctor (analogous to m) preserving finite products $(\times)$ on a cartesian-closed category (H).

A two-cover model $\mathcal{M} = (C, \blacktriangleleft, V)$ of DCK$_\Box$ consists of a two-cover system $(C, \blacktriangleleft)$ and a valuation function V, where the modal covering relation $\blacktriangleleft$ satisfies, in addition to the usual modal refinability and localization conditions, the modal *seriality* and *confluence* conditions stated below:

- *Modal Seriality*: For all $w \in W$, there exists an α such that $w \blacktriangleleft \alpha$
- *Modal Confluence*: If $w \blacktriangleleft \alpha$ and $w \blacktriangleleft \beta$, then there exists a γ s.t. $w \blacktriangleleft \gamma$ and $\alpha \preceq \gamma \succeq \beta$

The truth of CK$_\Box$ formulas is defined as before ensuring $|\Box A|^{\mathcal{M}} = \langle\blacktriangleleft\rangle |A|^{\mathcal{M}}$, by extending the definition of the satisfaction relation for IPL with the following case for modal formulas:

$$\mathcal{M}, w \Vdash \Box A \text{ iff } \exists \beta.\, w \blacktriangleleft \beta \text{ and } \forall v \in \beta.\, \mathcal{M}, v \Vdash A$$

Entailment in a two-cover model $\mathcal{M}$ of DCK$_\Box$ is defined by incorporating dual contexts as $\Delta ; \Gamma \models_{\mathcal{M}} A$ if and only if $\langle\blacktriangleleft\rangle|\Delta|^{\mathcal{M}} \cap |\Gamma|^{\mathcal{M}} \subseteq |A|^{\mathcal{M}}$. The application of the modal operator $\langle\blacktriangleleft\rangle$ to the interpretation of the global context Δ ensures that all the assumptions in Δ are all valid. Continuing a previous convention, we will write $\Delta\,;\,\Gamma \models A$ to mean $\Delta\,;\,\Gamma \models_{\mathcal{M}} A$ for all models $\mathcal{M}$.

A two-cover model $\mathcal{M} = (C, \blacktriangleleft, V)$ of $\mathrm{DCK}_\Box$ determines an equivalent algebraic model $\widehat{\mathcal{M}} = (\widehat{C}, \langle\blacktriangleleft\rangle, V)$ of $\mathrm{DCK}_\Box$ such that $|A|^{\mathcal{M}} = [\![A]\!]^{\widehat{\mathcal{M}}}$ and $|\Gamma|^{\mathcal{M}} = [\![\Gamma]\!]^{\widehat{\mathcal{M}}}$ for all formulas A and contexts Γ in $\mathrm{CK}_\Box$. Recollect that the maximal element of the Heyting algbra $\widehat{C}$ is W and its meets are given by the intersection $\cap$ of localized-up sets. The modal seriality and confluence condition respectively allow us to show that the operator $\langle\blacktriangleleft\rangle$ satisfies the equations $W = \langle\blacktriangleleft\rangle W$ and $\langle\blacktriangleleft\rangle X \cap \langle\blacktriangleleft\rangle Y = \langle\blacktriangleleft\rangle(X \cap Y)$ desired of a $\mathrm{CK}_\Box$ algebra. The inequalities $\langle\blacktriangleleft\rangle W \subseteq W$ and $\langle\blacktriangleleft\rangle(X \cap Y) \subseteq \langle\blacktriangleleft\rangle X \cap \langle\blacktriangleleft\rangle Y$ hold readily since W is maximal and $\langle\blacktriangleleft\rangle$ is monotonic. The converse $W \subseteq \langle\blacktriangleleft\rangle W$ of the former follows from the seriality condition: any $w \in W$ has some modal cover $\alpha \subseteq W$ and thus we also have $w \in \langle\blacktriangleleft\rangle W$. Similarly for any $X, Y \in \mathcal{LU}(W)$, the inequality $\langle\blacktriangleleft\rangle X \cap \langle\blacktriangleleft\rangle Y \subseteq \langle\blacktriangleleft\rangle(X \cap Y)$ follows from the confluence condition: if $w \in \langle\blacktriangleleft\rangle X \cap \langle\blacktriangleleft\rangle Y$, then for some α, β we have $w \blacktriangleleft \alpha \subseteq X$ and $w \blacktriangleleft \beta \subseteq Y$, from which we obtain a cover γ of w s.t. $\alpha \preceq \gamma \succeq \beta$ by applying confluence. Since X and Y are up-sets, it must be the case that $\gamma \subseteq X$ and $\gamma \subseteq Y$, which means $w \blacktriangleleft \gamma \subseteq (X \cap Y)$ and thus $w \in \langle\blacktriangleleft\rangle(X \cap Y)$. Altogether we have shown the desired equalities.

To prove completeness for $\mathrm{DCK}_\Box$, we define a two-cover system by taking pairs of contexts, i.e. the set $\mathit{Ctx} \times \mathit{Ctx}$, for worlds, point-wise context inclusion for the preorder relation, and the below inductively defined relation $\blacktriangleleft_{\mathrm{DCK}_\Box}$ for the modal covering relation. The local covering relation is given by re-defining the relation $\lhd_{\mathrm{IPL}}$ for dual-contexts by prepending a global context Δ uniformly to all the cases.

$$\Delta\,;\,\Gamma \blacktriangleleft_{\mathrm{DCK}_\Box} \{\Delta\,;\,\cdot\} \qquad \frac{\Delta\,;\,\Gamma \in \mathit{Ant}\,(\Box A) \qquad \Delta, A\,;\,\Gamma \blacktriangleleft_{\mathrm{DCK}_\Box} \alpha}{\Gamma \blacktriangleleft_{\mathrm{DCK}_\Box} \alpha}$$

$$\frac{\Delta\,;\,\Gamma \in \mathit{Ant}\,(\bot)}{\Delta\,;\,\Gamma \blacktriangleleft_{\mathrm{DCK}_\Box} \emptyset} \qquad \frac{\Delta\,;\,\Gamma \in \mathit{Ant}\,(A \vee B) \qquad \Delta\,;\,\Gamma, A \blacktriangleleft_{\mathrm{DCK}_\Box} \alpha_1 \qquad \Delta\,;\,\Gamma, B \blacktriangleleft_{\mathrm{DCK}_\Box} \alpha_2}{\Delta\,;\,\Gamma \blacktriangleleft_{\mathrm{DCK}_\Box} \alpha_1 \cup \alpha_2}$$

Theorem 4.7 (Soundness and Completeness for $\mathrm{CK}_\Box$) $\Delta\,;\,\Gamma \vdash A$ *if and only if* $\Delta\,;\,\Gamma \models A$.

Proof. Using the above arguments, by re-establishing the truth lemma once again for $\mathrm{DCK}_\Box$. □

5 Discussion and Further Work

We have presented two-cover semantics as a conservative extension of Goldblatt's relational cover semantics for IMLs and shown as examples four IMLs which can be modeled using two-cover semantics. We have shown that two-cover semantics semantics retains the simplicity of model construction in Kripke-style semantics, while overcoming its reliance on classical reasoning to prove completeness.

Formalization in type theory. The results in this article have been formalized in the proof assistant and dependently-typed programming language Agda [2], whose underlying core type theory is constructive. Formalizing our results in Agda ensures that our results are indeed constructive and do not accidentally rely upon classical reasoning principles. To encode cover models in type theory, we use a type $X : \mathit{Type}$ in place of a set X and values $x : X$ in place of elements $x \in X$. We encode subsets $X, Y \subseteq W$ as functions $X, Y : W \to \mathit{Type}$, and the inclusion $X \subseteq Y$ as a function $\forall w.\, X w \to Y w$. The covering relation $\lhd \subseteq W \times \mathcal{P}(W)$ is decomposed into a neighborhood "directory" $N : W \to \mathit{Type}$ and a membership relation $\in\, : \forall w.\, W \to N\, w \to \mathit{Type}$. A cover $w \lhd \alpha$ is encoded by an element $\alpha : N\, w$, where a world $v : W$ in our encoding satisfies the relation $v \in \alpha$ if and only if there exists a world $v \in W$ such that $v \in \alpha$. We refer the reader to the accompanying formalization in Agda for examples and further details.

Normalization. The completeness proofs in the previous sections can be readily refined to give normalization algorithms for proofs in the natural deduction systems of the respective logics. The definition of normal and neutral forms for this purpose can be found in Appendix A, where each inference rule has been carefully defined to satisfy the subformula property. The normalization algorithms are implemented using the technique of Normalization by Evaluation, and can be found in the accompanying Agda formalization.

Theorem 5.1 (Normalization) *Every judgment derivable in the proof system for CM/SL/PLL/$\mathrm{CK}_\Box$ has a derivation in normal form. Moreover, every derivation can be normalized to one in normal form.*

Proof. By refining the statement of the truth lemma, for example in Lemma 2.9 for CM, as follows: the tuple $\mathcal{N} = (C_{\text{CM}}, \mathit{Nf})$ is a cover model for CM such that for every formula A, we have $|A|^{\mathcal{N}} \subseteq \mathit{Nf}\,(A)$ and $\mathit{Ne}\,(A) \subseteq |A|^{\mathcal{N}}$, where $\mathit{Nf}\,(A) = \{\Gamma \in \mathit{Ctx} \mid \Gamma \vdash_{\text{NF}} A\}$ and $\mathit{Ne}\,(A) = \{\Gamma \in \mathit{Ctx} \mid \Gamma \vdash_{\text{NE}} A\}$. □

Intuitionistic neighbourhood semantics. A body of work that is closely related to our approach is *neighborhood semantics* for intuitionistic modal logics [5,20,19,22]. The modalities $\Box$ and $\Diamond$ are modeled using a neighborhood function $\mathcal{N} : W \to \mathcal{P}(\mathcal{P}(W))$, for example in [19, Definition 4.1], as follows:

$$\mathcal{M}, w \Vdash \Box A \text{ iff } \forall w'.\, w \sqsubseteq w' \text{ implies } \exists \alpha.\, \alpha \in \mathcal{N}(w') \text{ and } \forall v.\, v \in \alpha \text{ implies } \mathcal{M}, v \Vdash A$$
$$\mathcal{M}, w \Vdash \Diamond A \text{ iff } \forall w'.\, w \sqsubseteq w' \text{ implies } \forall \alpha.\, \alpha \in \mathcal{N}(w') \text{ implies } \exists v.\, v \in \alpha \text{ and } \mathcal{M}, v \Vdash A$$

While these clauses can be presented equivalently using a modal covering relation $\blacktriangleleft \subseteq W \times \mathcal{P}(W)$, a key difference is that we have used a clause resembling the former to model *all* modalities, including $\Box$ and $\Diamond$, alike. Moreover, another difference is the treatment of the positive connectives in these works. They do not use a local covering relation $\lhd$ and instead follow the usual Kripke-style approach as follows:

$$\mathcal{M}, w \Vdash \bot \text{ iff false} \qquad\qquad \mathcal{M}, w \Vdash A \vee B \text{ iff } \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B$$

The completeness proofs (c.f. [19, Lemma 4.4]) rely on prime sets as a result, and are thus not constructive.

Further Work. In this article, our focus has been on IMLs with a single modality. Following Goldblatt's work on multi-modal logics [28, Section 7], it should be possible to extend two-cover semantics to logics such as CK and CS4, featuring both the $\Box$ and $\Diamond$ modalities, and Fitch-style formulations [13,15] that extend the logics $\text{CK}_{\Box}$ and $\text{CS4}_{\Box}$ with an additional modality 🔒.

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A Appendix

A.1 Proof of Proposition 2.2: Soundness for IPL

We show $\Gamma \models A$ by induction on the given derivation of $\Gamma \vdash A$. The interesting cases are:

- *Rule* $\Rightarrow$*-Intro*: We must show $|\Gamma| \subseteq |A \Rightarrow B|$, which states that for all $w \in |\Gamma|$ and all $w' \sqsupseteq w$, we have $w' \in |A|$ implies $w' \in |B|$. By applying Lemma 2.1 to Γ we know that $|\Gamma|$ is an up-set, and thus $w' \in |\Gamma|$. Since $w' \in |\Gamma|$ and $w' \in |A|$, we also have $w' \in |\Gamma, A|$. By the induction hypothesis $|\Gamma, A| \subseteq |B|$ we thus have $w' \in |B|$ as desired.
- *Rule* $\bot$*-Elim*: We must show $|\Gamma| \subseteq |A|$. Suppose some $w \in |\Gamma|$. From the IH $|\Gamma| \subseteq |\bot|$, we know $w \in |\bot|$, which means $w \lhd \emptyset \subseteq |A|$. By applying Lemma 2.1 to A, we know $|A|$ satisfies localization, and thus it must be case that $w \in |A|$.
- *Rule* $\vee$*-Elim*: We must show $|\Gamma| \subseteq |C|$ from the induction hypotheses $|\Gamma| \subseteq |A \vee B|$ (IH.1), $|\Gamma, A| \subseteq |C|$ (IH.2) and $|\Gamma, B| \subseteq |C|$ (IH.3). Suppose some $w \in |\Gamma|$. From IH.1, we known $w \in |A \vee B|$, which means all members of some cover $\alpha \rhd w$ are either in $|A|$ or $|B|$. Consider an arbitrary $v \in \alpha$. The reachability condition ensures that v refines w. By applying Lemma 2.1 to Γ, we know $|\Gamma|$ is an up-set, which means $v \in |\Gamma|$. If $v \in |A|$, then $v \in |\Gamma, A|$ and thus $v \in |C|$ by IH.2. Otherwise $v \in |B|$, then $v \in |\Gamma, B|$ and thus $v \in |C|$ by IH.2. As a result, any $v \in \alpha$ is in $|C|$, meaning $w \lhd \alpha \subseteq |C|$. By applying Lemma 2.1 to C, we know $|C|$ satisfies localization, and thus $w \in |C|$ as desired.

A.2 Proof-system for IPL

IPL/HYP
$$\frac{A \in \Gamma}{\Gamma \vdash A}$$

IPL/$\top$-INTRO
$$\Gamma \vdash \top$$

IPL/$\bot$-ELIM
$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}$$

IPL/$\wedge$-INTRO
$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

IPL/$\wedge$-ELIM-1
$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}$$

IPL/$\wedge$-ELIM-2
$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

IPL/$\Rightarrow$-INTRO
$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

IPL/$\Rightarrow$-ELIM
$$\frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

IPL/$\vee$-INTRO-1
$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$

IPL/$\vee$-INTRO-2
$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

IPL/$\vee$-ELIM
$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

$$\text{IPL/NE/Hyp}\quad \frac{A \in \Gamma}{\Gamma \vdash_{\text{NE}} A}$$

$$\text{IPL/NF/}\top\text{-Intro}\quad \Gamma \vdash_{\text{NF}} \top$$

$$\text{IPL/NF/}\bot\text{-Elim}\quad \frac{\Gamma \vdash_{\text{NE}} \bot}{\Gamma \vdash_{\text{NF}} A}$$

$$\text{IPL/NF/}\wedge\text{-Intro}\quad \frac{\Gamma \vdash_{\text{NF}} A \qquad \Gamma \vdash_{\text{NF}} B}{\Gamma \vdash_{\text{NF}} A \wedge B}$$

$$\text{IPL/NE/}\wedge\text{-Elim-1}\quad \frac{\Gamma \vdash_{\text{NE}} A \wedge B}{\Gamma \vdash_{\text{NE}} A}$$

$$\text{IPL/NE/}\wedge\text{-Elim-2}\quad \frac{\Gamma \vdash_{\text{NE}} A \wedge B}{\Gamma \vdash_{\text{NE}} B}$$

$$\text{IPL/NF/}\Rightarrow\text{-Intro}\quad \frac{\Gamma, A \vdash_{\text{NF}} B}{\Gamma \vdash_{\text{NF}} A \Rightarrow B}$$

$$\text{IPL/NE/}\Rightarrow\text{-Elim}\quad \frac{\Gamma \vdash_{\text{NE}} A \Rightarrow B \qquad \Gamma \vdash_{\text{NF}} A}{\Gamma \vdash_{\text{NE}} B}$$

$$\text{IPL/NF/}\vee\text{-Intro-1}\quad \frac{\Gamma \vdash_{\text{NF}} A}{\Gamma \vdash_{\text{NF}} A \vee B}$$

$$\text{IPL/NF/}\vee\text{-Intro-2}\quad \frac{\Gamma \vdash_{\text{NF}} B}{\Gamma \vdash_{\text{NF}} A \vee B}$$

$$\text{IPL/NF/}\vee\text{-Elim}\quad \frac{\Gamma \vdash_{\text{NE}} A \vee B \qquad \Gamma, A \vdash_{\text{NF}} C \qquad \Gamma, B \vdash_{\text{NF}} C}{\Gamma \vdash_{\text{NF}} C}$$

A.3 Proof-systems for the logics CM, SL and PLL

$$\text{CM/}\heartsuit\text{-Mon}\quad \frac{\Gamma \vdash \heartsuit A \qquad A \vdash B}{\Gamma \vdash \heartsuit B}$$

$$\text{SL/}\Diamond\text{-Map}\quad \frac{\Gamma \vdash \Diamond A \qquad \Gamma, A \vdash B}{\Gamma \vdash \Diamond B}$$

$$\text{PLL/}\Diamond\text{-Intro}\quad \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}$$

$$\text{PLL/}\Diamond\text{-Bind}\quad \frac{\Gamma \vdash \Diamond A \qquad \Gamma, A \vdash \Diamond B}{\Gamma \vdash \Diamond B}$$

$$\begin{aligned} \text{CM} &:= \text{IPL} + \text{CM/}\heartsuit\text{-Mon} \\ \text{SL} &:= \text{IPL} + \text{SL/}\Diamond\text{-Map} \\ \text{PLL} &:= \text{IPL} + \text{PLL/}\Diamond\text{-Intro} + \text{PLL/}\Diamond\text{-Bind} \end{aligned}$$

$$\text{CM/NF/}\heartsuit\text{-Mon}\quad \frac{\Gamma \vdash_{\text{NE}} \heartsuit A \qquad A \vdash_{\text{NF}} B}{\Gamma \vdash_{\text{NF}} \heartsuit B}$$

$$\text{SL/NF/}\Diamond\text{-Map}\quad \frac{\Gamma \vdash_{\text{NE}} \Diamond A \qquad \Gamma, A \vdash_{\text{NF}} B}{\Gamma \vdash_{\text{NF}} \Diamond B}$$

$$\text{PLL/NF/}\Diamond\text{-Intro}\quad \frac{\Gamma \vdash_{\text{NF}} A}{\Gamma \vdash_{\text{NF}} \Diamond A}$$

$$\text{PLL/NF/}\Diamond\text{-Bind}\quad \frac{\Gamma \vdash_{\text{NE}} \Diamond A \qquad \Gamma, A \vdash_{\text{NF}} \Diamond B}{\Gamma \vdash_{\text{NF}} \Diamond B}$$

A.4 Proof-system $DCK_\Box$ for the logic $CK_\Box$

$$\text{DCK}_\Box\text{/Hyp}\quad \frac{A \in \Gamma}{\Delta\,;\Gamma \vdash A}$$

$$\text{DCK}_\Box\text{/}\top\text{-Intro}\quad \Delta\,;\Gamma \vdash \top$$

$$\text{DCK}_\Box\text{/}\bot\text{-Elim}\quad \frac{\Delta\,;\Gamma \vdash \bot}{\Delta\,;\Gamma \vdash A}$$

$$\text{DCK}_\Box\text{/}\wedge\text{-Intro}\quad \frac{\Delta\,;\Gamma \vdash A \qquad \Delta\,;\Gamma \vdash B}{\Delta\,;\Gamma \vdash A \wedge B}$$

$$\text{DCK}_\Box\text{/}\wedge\text{-Elim-1}\quad \frac{\Delta\,;\Gamma \vdash A \wedge B}{\Delta\,;\Gamma \vdash A}$$

$$\text{DCK}_\Box\text{/}\wedge\text{-Elim-2}\quad \frac{\Delta\,;\Gamma \vdash A \wedge B}{\Delta\,;\Gamma \vdash B}$$

$$\text{DCK}_\Box\text{/}\Rightarrow\text{-Intro}\quad \frac{\Delta\,;\Gamma, A \vdash B}{\Delta\,;\Gamma \vdash A \Rightarrow B}$$

$$\text{DCK}_\Box\text{/}\Rightarrow\text{-Elim}\quad \frac{\Delta\,;\Gamma \vdash A \Rightarrow B \qquad \Delta\,;\Gamma \vdash A}{\Delta\,;\Gamma \vdash B}$$

$$\text{DCK}_\Box\text{/}\vee\text{-Intro-1}\quad \frac{\Delta\,;\Gamma \vdash A}{\Delta\,;\Gamma \vdash A \vee B}$$

$$\text{DCK}_\Box\text{/}\vee\text{-Intro-2}\quad \frac{\Delta\,;\Gamma \vdash B}{\Delta\,;\Gamma \vdash A \vee B}$$

$$\text{DCK}_\Box\text{/}\vee\text{-Elim}\quad \frac{\Delta\,;\Gamma \vdash A \vee B \qquad \Delta\,;\Gamma, A \vdash C \qquad \Delta\,;\Gamma, B \vdash C}{\Gamma \vdash C}$$

$$\text{DCK}_\Box\text{/}\Box\text{-Intro}\quad \frac{\Delta\,;\cdot \vdash A}{\Delta\,;\Gamma \vdash \Box A}$$

$$\text{DCK}_\Box\text{/}\Box\text{-Elim}\quad \frac{\Delta\,;\Gamma \vdash \Box A \qquad \Delta, A\,;\Gamma \vdash \Box B}{\Delta\,;\Gamma \vdash B}$$

$$\textsc{DCK}_\Box\text{/NE/}\textsc{Hyp}\quad \frac{A \in \Gamma}{\Delta\,;\,\Gamma \vdash_{\text{NE}} A} \qquad \textsc{DCK}_\Box\text{/NF/}\top\text{-}\textsc{Intro}\quad \Delta\,;\,\Gamma \vdash_{\text{NF}} \top$$

$$\textsc{DCK}_\Box\text{/NF/}\bot\text{-}\textsc{Elim}\quad \frac{\Delta\,;\,\Gamma \vdash_{\text{NE}} \bot}{\Delta\,;\,\Gamma \vdash_{\text{NF}} A} \qquad \textsc{DCK}_\Box\text{/NF/}\wedge\text{-}\textsc{Intro}\quad \frac{\Delta\,;\,\Gamma \vdash_{\text{NF}} A \qquad \Delta\,;\,\Gamma \vdash_{\text{NF}} B}{\Delta\,;\,\Gamma \vdash_{\text{NF}} A \wedge B}$$

$$\textsc{DCK}_\Box\text{/NE/}\wedge\text{-}\textsc{Elim-1}\quad \frac{\Delta\,;\,\Gamma \vdash_{\text{NE}} A \wedge B}{\Delta\,;\,\Gamma \vdash_{\text{NE}} A} \qquad \textsc{DCK}_\Box\text{/NE/}\wedge\text{-}\textsc{Elim-2}\quad \frac{\Delta\,;\,\Gamma \vdash_{\text{NE}} A \wedge B}{\Delta\,;\,\Gamma \vdash_{\text{NE}} B}$$

$$\textsc{DCK}_\Box\text{/NF/}\Rightarrow\text{-}\textsc{Intro}\quad \frac{\Delta\,;\,\Gamma, A \vdash_{\text{NF}} B}{\Delta\,;\,\Gamma \vdash_{\text{NF}} A \Rightarrow B} \qquad \textsc{DCK}_\Box\text{/NE/}\Rightarrow\text{-}\textsc{Elim}\quad \frac{\Delta\,;\,\Gamma \vdash_{\text{NE}} A \Rightarrow B \qquad \Delta\,;\,\Gamma \vdash_{\text{NF}} A}{\Delta\,;\,\Gamma \vdash_{\text{NE}} B}$$

$$\textsc{DCK}_\Box\text{/NF/}\vee\text{-}\textsc{Intro-1}\quad \frac{\Delta\,;\,\Gamma \vdash_{\text{NF}} A}{\Delta\,;\,\Gamma \vdash_{\text{NF}} A \vee B} \qquad \textsc{DCK}_\Box\text{/NF/}\vee\text{-}\textsc{Intro-2}\quad \frac{\Delta\,;\,\Gamma \vdash_{\text{NF}} B}{\Delta\,;\,\Gamma \vdash_{\text{NF}} A \vee B}$$

$$\textsc{DCK}_\Box\text{/NF/}\vee\text{-}\textsc{Elim}\quad \frac{\Delta\,;\,\Gamma \vdash_{\text{NE}} A \vee B \qquad \Delta\,;\,\Gamma, A \vdash_{\text{NF}} C \qquad \Delta\,;\,\Gamma, B \vdash_{\text{NF}} C}{\Gamma \vdash_{\text{NF}} C}$$

$$\textsc{DCK}_\Box\text{/NF/}\Box\text{-}\textsc{Intro}\quad \frac{\Delta\,;\,\cdot \vdash_{\text{NF}} A}{\Delta\,;\,\Gamma \vdash_{\text{NF}} \Box A} \qquad \textsc{DCK}_\Box\text{/NF/}\Box\text{-}\textsc{Elim}\quad \frac{\Delta\,;\,\Gamma \vdash_{\text{NE}} \Box A \qquad \Delta, A\,;\,\Gamma \vdash_{\text{NF}} \Box B}{\Delta\,;\,\Gamma \vdash_{\text{NF}} B}$$