Kripke-Style Semantics for Strong Functors

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Abstract. Strong functors are a ubiquitous programming abstraction in typed-functional programming. Strong functors originate from category theory, and the categorical semantics of strong functors in typed-lambda calculi are thus naturally well-understood. The same cannot be said, however, of their Kripke-style or possible-world semantics, which are of recurring interest in formal logic and programming languages. We address this gap by identifying possible-world semantics for a family of lambda calculi featuring strong functors. We then study the connection to categorical semantics and illustrate an application of this connection in modular implementation and verification of normalization algorithms.

Keywords: intuitionistic modal logic · presheaves · normalization

1 Introduction

In typed-functional programming *functors*, or precisely *strong functors*, are a programming abstraction that capture the essence of mapping a function over a data structure while preserving its structure. For instance, in the programming language Haskell, a type constructor Arr_n that denotes arrays of size n is an example of a strong functor. Over an array of type $\mathsf{Arr}_n\, A$ we may map a function $f : A \Rightarrow B$ to obtain an array of type $\mathsf{Arr}_n\, B$. Arrays, heaps, lists, trees are some examples of several data structures that can be viewed as strong functors. *Monads*, or precisely *strong monads*, are a special class of strong functors that are popular for modelling effects as data in a pure functional programming language [26,33]. Despite the widespread application of strong functors and monads, there remain some pressing open questions in their theoretical foundations.

Strong functors and monads originate from category theory, and their computational behavior in lambda calculi have been derived from their categorical counterparts. Given the direct correspondence, categorical semantics of strong functors in lambda calculi are well-understood [18,26,24]. For the study of certain meta-theoretic properties, such as lambda definability and normalization, however, categorical semantics alone is insufficient. The focus of categorical semantics (in this area) has been that of generality, and it does not restrict our attention to a class of models small enough for a specific purpose. Constructing a categorical model thus requires a significant amount of ingenuity even when it need not. A complementary approach that has shown promise to alleviate this difficulty is the so-called Kripke-style or *possible-world* semantics [25].

Possible-world semantics originate from formal logic and were famously used by Saul Kripke to study the completeness of classical modal logic [19] and intuitionistic propositional logic [20]. Mitchell and Moggi [25] later introduced possible-world semantics (dubbed Kripke-style semantics) to typed-lambda calculi for "their practical advantage" of being "easy to devise Kripke counter-models". They note that possible-world semantics "seem to support a set-like intuition about lambda terms better than arbitrary cartesian closed categories". They also put forth *Kripke logical relations*, which have become standard equipment to prove meta-theoretic properties about lambda calculi in a variety of domains.

One area where the study of possible-world semantics appears to be rewarding (and has led to renewed interest [31]) is normalization. Normalization is a valuable meta-theoretic property that is difficult to prove. Catarina Coquand [10,11] proved normalization for a typed lambda calculus in the proof assistant Alf [23] by constructing a possible-world model as an instance of Mitchell and Moggi's semantics. This model-based approach to normalization, known as Normalization by Evaluation (NbE) [8,7], dispenses with tedious syntactic reasoning that typically complicate normalization proofs and is amenable to mechanization in a proof assistant. A notable corollary in Coquand's work is the constructive proof of completeness for possible-world semantics that follows from normalization.

Our work can be seen as an extension of the work of Mitchell and Moggi [25] and Coquand [11] to lambda calculi with strong functors. We develop possible-world semantics for a family of lambda calculi with strong functors à la Mitchell and Moggi and, akin to Coquand, mechanize completeness proofs for each calculus by constructing NbE models in Agda. In the process, we identify a suitable extension of Kripke logical relations for these calculi to prove normalization.

The development of possible-world semantics for strong functors has also been of interest from a logical perspective as a foundation for type systems [6,4]. Viewed through the lens of the Curry-Howard correspondence, a strong functor type can be understood as a modality in the underlying intuitionistic logic. In this view, our work addresses the following open questions:

1. What is the intuitionistic modal logic that corresponds to strong functors?
2. How do we give possible-world semantics to this modal logic?
3. How do the possible-world semantics of this modal logic relate to the computational behaviour of strong functors and their categorical semantics?

If we restrict our attention in the above questions to strong monads, Fairtlough and Mendler [13] and Benton, Bierman and de Paiva [6] answer the first two, and Alechina et al. [4] provide a partial answer to the third. A satisfactory treatment of question 3 that demonstrates a link through presheaves is still missing. We address this gap, not just for strong monads, but also for weaker strong functors.

Normalization for strong monads—specifically extensions of Moggi's monadic meta-language [26]—is well-understood [14,21,3], but has not been studied from the perspective of possible-world semantics or modal logic. The creative contribution of this article lies in its observation that normalization algorithms for strong functors (some known and new) can be constructed systematically as instances of possible-world semantics by doing lesser "routine work".

The family of calculi we study in this article are minimal extensions of the simply-typed lambda calculus (STLC) with the following operations or *axioms*:

$$\mathrm{S} : A \times \Diamond B \Rightarrow \Diamond(A \times B) \qquad \mathrm{R} : A \Rightarrow \Diamond A \qquad \mathrm{J} : \Diamond\Diamond A \Rightarrow \Diamond A$$

We define a calculus λ_{SF} for strong functors by extending STLC with a unary type constructor $\Diamond$ that admits axiom S (for "strength"). By further extending λ_{SF} with the axioms R (for "return") and J (for "join"), we arrive at three more calculi: a calculus for strong *pointed* functors λ_{PF} that admits axiom S and R (but not J), a calculus for strong *semimonads* (or *joinable* functors) λ_{JF} that admits axiom S and J (but not R), and a calculus for strong monads λ_{ML} (also known as Moggi's monadic metalanguage [26]) that admits all three axioms[1].

In Section 2, we restrict our attention to the calculus λ_{SF}. We present a simplified possible-world semantics that interprets the strong functor type former $\Diamond$ as an intuitionistic possibility modality, and illustrate the construction of an NbE model as an instance. In Section 3, we use a refined *two-dimensional* possible-world semantics [17], which can also be called presheaf semantics, that makes the connection to categorical semantics evident and provides an opportunity for modular construction of NbE models. In Section 4, we show that the two-dimensional approach extends seamlessly to the remaining calculi λ_{PF}, λ_{JF}, and λ_{ML}. In Section 5 we discuss related work on modal logic and lambda calculi, and conclude with observations about possible extensions and current limitations. All the key formal results in this article have been mechanized in Agda[2].

2 Possible-World Models of Strength

The types and typing contexts of λ_{SF}, λ_{PF}, λ_{JF}, and λ_{ML} are defined alike as:

$$\mathsf{Ty} \qquad A, B := \iota \mid \top \mid A \times B \mid A \Rightarrow B \mid \Diamond A \qquad \mathsf{Ctx} \qquad \Gamma, \Delta := \cdot \mid \Gamma, A$$

The type ι denotes an uninterpeted base type (i.e., a ground type with no specific operations), $\top$ denotes the unit type, $A \times B$ denotes product types, $A \Rightarrow B$ denotes function types, and $\Diamond A$ denotes strong functor types. The context $\cdot$ denotes the empty context, while Γ, A denotes the extension of a context Γ with a type A.

In this section, we define the terms, typing rules and equational theory of λ_{SF} and observe its categorical semantics. We present a simplified possible-world semantics for λ_{SF} that ignores the equational theory of λ_{SF} and views λ_{SF} as a natural deduction proof calculus for judgements $\Gamma \vdash A$. We then illustrate the construction of an NbE model for λ_{SF} as an instance of this semantics. We conclude this section by observing a difficulty in adapting the simplified semantics to incorporate the equational theory of λ_{SF}, and the overhead of repeating this process for the remaining calculi λ_{PF}, λ_{JF}, and λ_{ML}. In the later Sections 3 and 4, we address this difficulty using two-dimensional possible-world semantics.

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[1] Observe that axiom S is interderivable with its alternative formulation in functional programming as a function $\mathsf{fmap} : (A \Rightarrow B) \Rightarrow \Diamond A \Rightarrow \Diamond B$, while the axioms R and J are immediately the monadic functions $\mathsf{return} : A \Rightarrow \Diamond A$ and $\mathsf{join} : \Diamond\Diamond A \Rightarrow \Diamond A$.

[2] `https://github.com/nachivpn/s`

2.1 The calculus λ_{SF}

The terms, typing rules and equational theory of λ_{SF} are defined in Figure 1. The judgements $\Gamma \vdash t : A$ define intrinsically well-typed terms of λ_{SF} and judgements $\Gamma \vdash t \sim t' : A$ define well-typed equations. We define well-typed (and scoped) variables using de Bruijn indices as judgments $\Gamma \vdash_{\mathrm{VAR}} v : A$ with constructs zero and succ. The notation $t[u]$ denotes the substitution of term u in t for the variable zero, and the operator wk "weakens" a term $\Gamma \vdash t : A$ by embedding it into a larger context $\Gamma \leq \Gamma'$ as $\Gamma' \vdash \mathit{wk}\, t : A$.

The λ_{SF} calculus extends STLC (featuring products) with a construct $\mathsf{letmap}_{\mathrm{SF}}$ (see Rule SF/$\Diamond$-Letmap) that "maps" a term $\Gamma, A \vdash u : B$ over a term $\Gamma \vdash t : \Diamond A$ and two new equations (Rule SF/$\Diamond$-η and Rule SF/$\Diamond$-β).

$$\textsc{Var-Zero}\quad \Gamma, A \vdash_{\mathrm{VAR}} \mathsf{zero} : A \qquad \textsc{Var-Succ}\quad \frac{\Gamma \vdash_{\mathrm{VAR}} v : A}{\Gamma, B \vdash_{\mathrm{VAR}} \mathsf{succ}\, v : A} \qquad \textsc{Var}\quad \frac{\Gamma \vdash_{\mathrm{VAR}} v : A}{\Gamma \vdash \mathsf{var}\, v : A}$$

$$\top\textsc{-Intro}\quad \Gamma \vdash \mathsf{unit} : \top \qquad \times\textsc{-Intro}\quad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash \mathsf{pair}\, t\, u : A \times B} \qquad \times\textsc{-Elim-1}\quad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathsf{fst}\, t : A} \qquad \times\textsc{-Elim-2}\quad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathsf{snd}\, t : B}$$

$$\Rightarrow\textsc{-Intro}\quad \frac{\Gamma, A \vdash t : B}{\Gamma \vdash \lambda\, t : A \Rightarrow B} \qquad \Rightarrow\textsc{-Elim}\quad \frac{\Gamma \vdash t : A \Rightarrow B \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{app}\, t\, u : B} \qquad \mathrm{SF}/\Diamond\textsc{-Letmap}\quad \frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : B}{\Gamma \vdash \mathsf{letmap}_{\mathrm{SF}}\, t\, u : \Diamond B}$$

$$\top\text{-}\eta\quad \frac{\Gamma \vdash t : \top}{\Gamma \vdash t \sim \mathsf{unit} : \top} \qquad \times\text{-}\eta\quad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash t \sim \mathsf{pair}\, (\mathsf{fst}\, t)\, (\mathsf{snd}\, t) : A \times B} \qquad \times\text{-}\beta_1\quad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash \mathsf{fst}\, (\mathsf{pair}\, t\, u) \sim t : A}$$

$$\times\text{-}\beta_2\quad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash \mathsf{snd}\, (\mathsf{pair}\, t\, u) \sim u : B} \qquad \Rightarrow\text{-}\eta\quad \frac{\Gamma \vdash t : A \Rightarrow B}{\Gamma \vdash t \sim \lambda\, (\mathsf{app}\, (\mathit{wk}\, t)\, (\mathsf{var}\, \mathsf{zero})) : A \Rightarrow B}$$

$$\Rightarrow\text{-}\beta\quad \frac{\Gamma, A \vdash t : B \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{app}\, (\lambda\, t)\, u \sim t[u] : B} \qquad \mathrm{SF}/\Diamond\text{-}\eta\quad \frac{\Gamma \vdash t : \Diamond A}{\Gamma \vdash t \sim \mathsf{letmap}_{\mathrm{SF}}\, t\, (\mathsf{var}\, \mathsf{zero}) : \Diamond A}$$

$$\mathrm{SF}/\Diamond\text{-}\beta\quad \frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : B \qquad \Gamma, B \vdash u' : C}{\Gamma \vdash \mathsf{letmap}_{\mathrm{SF}}\, (\mathsf{letmap}_{\mathrm{SF}}\, t\, u)\, u' \sim \mathsf{letmap}_{\mathrm{SF}}\, t\, (u'[u]) : \Diamond C}$$

Fig. 1. Well-typed terms and equational theory for λ_{SF}

Categorical semantics A categorical model of λ_{SF} is given by a cartesian-closed category $\mathcal{C}$ $(1, \times, \Rightarrow)$ with a strong functor $\Diamond : \mathcal{C} \to \mathcal{C}$ and a $\mathcal{C}$-object V_ι that interprets the base type ι. Given a model $\mathcal{C}$ of λ_{SF}, we interpret types in λ_{SF} as $\mathcal{C}$-objects and terms $\Gamma \vdash t : A$ as $\mathcal{C}$-morphisms $[\![t]\!] : [\![\Gamma]\!] \to [\![A]\!]$ (at times written explicitly as $[\![\Gamma]\!] \to_{\mathcal{C}} [\![A]\!]$) by induction on types and terms respectively:

$$\begin{array}{ll}
[\![\iota]\!] = V_\iota & [\![-]\!] : \Gamma \vdash A \to ([\![\Gamma]\!] \to_{\mathcal{C}} [\![A]\!]) \\
[\![\top]\!] = 1 & [\![\ldots]\!] = \ldots \\
[\![A \times B]\!] = [\![A]\!] \times [\![B]\!] & [\![\mathsf{pair}\, t\, u]\!] = \langle [\![t]\!], [\![u]\!] \rangle \\
[\![A \Rightarrow B]\!] = [\![A]\!] \Rightarrow [\![B]\!] & [\![\ldots]\!] = \ldots \\
[\![\Diamond A]\!] = \Diamond [\![A]\!] & [\![\mathsf{letmap}_{\mathrm{SF}}\, t\, u]\!] = \Diamond [\![u]\!] \circ \theta_{[\![\Gamma]\!],[\![B]\!]} \circ \langle \mathit{id}_{[\![\Gamma]\!]}, [\![t]\!] \rangle
\end{array}$$

The STLC constructs are interpreted as usual with the cartesian-closure of $\mathcal{C}$. For example, we interpret a pair of terms $\mathsf{pair}\, t\, u$ by a pair of morphisms $\langle [\![t]\!], [\![u]\!] \rangle$. We define $[\![\mathsf{letmap}_{\mathrm{SF}}\, t\, u]\!] : [\![\Gamma]\!] \to \Diamond [\![A]\!]$ for terms $\Gamma \vdash t : \Diamond B$ and $\Gamma, B \vdash u : A$ by constructing a morphism $\langle \mathit{id}_{[\![\Gamma]\!]}, [\![t]\!] \rangle : [\![\Gamma]\!] \to [\![\Gamma]\!] \times \Diamond [\![B]\!]$, and composing it with $\theta_{[\![\Gamma]\!],[\![B]\!]} : [\![\Gamma]\!] \times \Diamond [\![B]\!] \to \Diamond([\![\Gamma]\!] \times [\![B]\!])$ and $\Diamond [\![u]\!] : \Diamond([\![\Gamma]\!] \times [\![B]\!]) \to \Diamond [\![A]\!]$, where the former is given by strength and the latter by functorial action of $\Diamond$.

Proposition 1 (Categorical semantics for λ_{SF}). *Given two terms t, u in λ_{SF}, $\Gamma \vdash t \sim u : A$ if and only if for all models $\mathcal{C}$ of λ_{SF} $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \to_{\mathcal{C}} [\![A]\!]$.*

Proof. Follows by induction on the judgment $\Gamma \vdash t \sim u : A$ in one direction, and by a term model construction (see for e.g., [9, Section 3.2]) in the converse.

2.2 Possible-world semantics

Meta-theory Traditionally, the possible-world semantics for a modal logic is given in a classical meta-theory using sets and relations. In contrast, we work in a constructive type-theoretic meta-theory that resembles that of Agda [1], and denote the universe of types in this language by *Type*. This means that we shall use a type W : *Type* in place of a set W, and correspondingly a relation on types $R : W \to W \to \mathit{Type}$ in place of a relation on sets $R \subseteq W \times W$.

Frames and Models A possible-world *frame* $F = (W, R_i, R_m)$ is a triple that consists of a type W : *Type* of *worlds* and two *accessibility* relations R_i, R_m: $W \to W \to \mathit{Type}$ (for "intuitionistic" and "modal") on worlds, subject to the *frame conditions* that R_i is reflexive and transitive and ${R_i}^{-1}; R_m \subseteq R_m; {R_i}^{-1}$, as witnessed by the functions:

- $\mathit{refl}_i : \forall w.\, w\ R_i\ w$
- $\mathit{trans}_i : \forall w, w', w''.\, w\ R_i\ w' \to w'\ R_i\ w'' \to w\ R_i\ w''$
- $\mathit{factor} : \forall w, w', v.\, w\ R_i\ w' \to w\ R_m\ v \to \exists v'.\, (w'\ R_m\ v' \times v\ R_i\ v')$

A possible-world *model* $M = (F, V)$ couples a frame F with a valuation function V that assigns to a base type ι a world-indexed family $V_\iota : W \to \mathit{Type}$ accompanied by a "weakening" function $\mathit{wk}_\iota : \forall w, w'.\, w\ R_i\ w' \to V_{\iota,w} \to V_{\iota,w'}$.

Interpreting Types Given a possible-world model $M = (F, V)$, the possible-world interpretation of types in λ_{SF} is given by interpreting a type A (in the calculus) as a family of types $[\![A]\!]_w : \mathit{Type}$ (in the meta-theory) indexed by worlds $w : W$.

$$\begin{aligned}
[\![\iota]\!]_w &= V_{\iota,w} \\
[\![\top]\!]_w &= \top \\
[\![A \times B]\!]_w &= [\![A]\!]_w \times [\![B]\!]_w \\
[\![A \Rightarrow B]\!]_w &= \forall w'.\, w\ R_i\ w' \to [\![A]\!]_{w'} \to [\![B]\!]_{w'} \\
[\![\Diamond A]\!]_w &= \exists v.\, w\ R_m\ v \times [\![A]\!]_v
\end{aligned}$$

The interpretation of the base type ι is given by the valuation function V, and the unit, product and function types are interpreted as usual in a possible-world model. We interpret the strong functor type former as a possibility modality in modal logic: the interpretation of a type $\Diamond A$ at a world w is given by the interpretation of A at some "future" world v along with a proof of $w\ R_m\ v$ witnessing the connection from w to v via R_m. The typing contexts are interpreted as usual by taking the cartesian product of families: $[\![\cdot]\!]_w = \top$, where $\top$ denotes the nullary product, and $[\![\Gamma, A]\!]_w = [\![\Gamma]\!]_w \times [\![A]\!]_w$ for some arbitrary world w.

A desired property of possible-world semantics, known as the *monotonicity* lemma, can be shown to be retained under the given interpretation of types.

Lemma 1 (Monotonicity lemma). *For all types A, we have a weakening function $\mathit{wk}_A : \forall w, w'.\, w\ R_i\ w' \to [\![A]\!]_w \to [\![A]\!]_{w'}$, and similarly for contexts, we have $\mathit{wk}_\Gamma : \forall w, w'.\, w\ R_i\ w' \to [\![\Gamma]\!]_w \to [\![\Gamma]\!]_{w'}$ for every context Γ.*

Intuitively, the monotonicity lemma allows us to "transport" along the relation R_i an element $a : [\![A]\!]_w$ at a world w, given $i : w\ R_i\ w'$, to an intuitionistic future w' as $\mathit{wk}\ i\ a : [\![A]\!]_{w'}$, . This lemma is proved by induction on types (and similarly contexts), where the case of $\Diamond A$ is dealt with using the frame condition *factor*.

Inclusion Condition To interpret terms of λ_{SF}, we must impose an additional "inclusion" condition $R_m \subseteq R_i$ on possible-world frames. This condition, in light of Lemma 1, enables us to transport elements along the relation R_m. From a logical perspective, this condition enables models to *validate* the characteristic axiom $\mathrm{S} : A \times \Diamond B \Rightarrow \Diamond(A \times B)$ of λ_{SF}. The following proposition explains how.

We say that a model M validates an axiom X, denoted $M \models X$, to mean that the interpretation $[\![X]\!]_w$ is inhabited (or "holds") for all worlds w in M and for all instantiations of axiom X's type scheme. We say that frame F validates axiom X, denoted $F \models X$, to mean that $(F, V) \models X$ for all valuations V.

Proposition 2. *Give any frame $F = (W, R_i, R_m)$, $F \models S$ if the underlying accessibility relations R_i and R_m satisfy the inclusion condition $R_m \subseteq R_i$, as witnessed by a function $\mathit{incl} : \forall w, v.\, w\ R_m\ v \to w\ R_i\ v$.*

Proof. For some world w, and types A, B, we must show $[\![A \times \Diamond B \Rightarrow \Diamond(A \times B)]\!]_w$. This amounts to showing $[\![A]\!]_{v'}$ and $[\![B]\!]_{v'}$ for some world v' such that $w'\ R_m\ v'$, given proofs $i : w\ R_i\ w'$, $m : w'\ R_m\ v$ and elements $a : [\![A]\!]_{w'}$ and $b : [\![B]\!]_v$ for worlds $w', v : W$. We pick v for v' and obtain $b : [\![B]\!]_{v'}$ and $m : w'\ R_m\ v'$. We then obtain an R_i-proof $\mathit{incl}\ m : w'\ R_i\ v'$ from m using *incl* and transport a to world v' by applying the montonicity lemma (Lemma 1) as $\mathit{wk}_A\ (\mathit{incl}\ m)\ a : [\![A]\!]_{v'}$.

Interpreting Terms A possible-world model of λ_{SF} is a possible-world model (F, V_ι) whose frame $F = (W, R_i, R_m)$ satisfies the inclusion condition $R_m \subseteq R_i$. The terms in λ_{SF} are interpreted as a family of functions as follows:

$$
\begin{aligned}
&[\![-]\!] : \Gamma \vdash A \to (\forall w.\, [\![\Gamma]\!]_w \to [\![A]\!]_w)\\
&[\![\mathsf{var}\, v]\!]\, \gamma = \mathit{lookup}\, v\, \gamma\\
&[\![\mathsf{unit}]\!]\, \gamma = ()\\
&[\![\mathsf{pair}\, t\, u]\!]\, \gamma = ([\![t]\!]\, \gamma, [\![u]\!]\, \gamma)\\
&[\![\mathsf{fst}\, t]\!]\, \gamma = \pi_1([\![t]\!]\, \gamma)\\
&[\![\mathsf{snd}\, t]\!]\, \gamma = \pi_2([\![t]\!]\, \gamma)\\
&[\![\lambda\, t]\!]\, \gamma = \lambda i.\, \lambda a.\, [\![t]\!]\, (\mathit{wk}_\Gamma\, i\, \gamma, a)\\
&[\![\mathsf{app}\, t\, u]\!]\, \gamma = ([\![t]\!]\, \gamma)\, \mathit{refl}_i\, ([\![u]\!]\, \gamma)\\
&[\![\mathsf{letmap}_{\mathrm{SF}}\, t\, u]\!]\, \gamma = (m, [\![u]\!]\, (\mathit{wk}_\Gamma\, (\mathit{incl}\, m)\, \gamma, a))\\
&\qquad \text{where } (m : w\ R_m\ v, a : [\![A]\!]_v) = [\![t]\!]\, \gamma
\end{aligned}
$$

Interpretation of STLC terms follows the usual routine: we interpret variables by projecting the environment $\gamma : [\![\Gamma]\!]_w$ using a function *lookup*, the unit and pair constructs (unit, pair, fst, snd) with their semantic counterparts ($()$, $(-,-)$, π_1, π_2), and the function constructs (λ,app) with semantic function abstraction and application while handling R_i-proofs appropriately. The interesting case is that of $\mathsf{letmap}_{\mathrm{SF}}$: given terms $\Gamma \vdash t : \Diamond A$ and $\Gamma, A \vdash u : B$, and an environment $\gamma : [\![\Gamma]\!]_w$, we must produce an element of type $[\![\Diamond B]\!]_w = \exists v.\, w\ R_m\ v \times [\![B]\!]_v$. Recursively interpreting t gives us a pair $(m : w\ R_m\ v, a : [\![A]\!]_v)$, using the former of which we transport γ along R_m to the world v, as $\mathit{wk}_\Gamma\, (\mathit{incl}\, m)\, \gamma : [\![\Gamma]\!]_v$, which is in turn used to recursively interpret u, thus obtaining the desired element of type $[\![B]\!]_v$.

Normalization by Evaluation The objective of normalization is to define a function $\mathit{norm} : \Gamma \vdash A \to \Gamma \vdash_{\mathrm{NF}} A$, assigning a normal form to every term in λ_{SF}. The judgements $\Gamma \vdash_{\mathrm{NF}} A$ defined in Figure 2 characterize normal forms. As usual, they are defined alongside judgements $\Gamma \vdash_{\mathrm{NE}} A$ denoting "neutral" terms.

$$
\frac{\Gamma \vdash_{\mathrm{VAR}} v : A}{\Gamma \vdash_{\mathrm{NE}} \mathsf{var}\, v : A}\ \textsc{Ne/Var}
\qquad
\frac{\Gamma \vdash_{\mathrm{NE}} n : \iota}{\Gamma \vdash_{\mathrm{NF}} \mathsf{up}\, n : \iota}\ \textsc{Nf/Up}
\qquad
\frac{}{\Gamma \vdash_{\mathrm{NF}} \mathsf{unit} : \top}\ \textsc{Nf/Unit}
\qquad
\frac{\Gamma \vdash_{\mathrm{NE}} n : A \times B}{\Gamma \vdash_{\mathrm{NE}} \mathsf{fst}\, n : A}\ \textsc{Ne/}\times\textsc{-Elim-1}
$$

$$
\frac{\Gamma \vdash_{\mathrm{NE}} n : A \times B}{\Gamma \vdash_{\mathrm{NE}} \mathsf{snd}\, n : B}\ \textsc{Ne/}\times\textsc{-Elim-2}
\qquad
\frac{\Gamma \vdash_{\mathrm{NF}} n : A \qquad \Gamma \vdash_{\mathrm{NF}} m : B}{\Gamma \vdash_{\mathrm{NF}} \mathsf{pair}\, n\, m : A \times B}\ \textsc{Nf/}\times\textsc{-Intro}
\qquad
\frac{\Gamma, A \vdash_{\mathrm{NF}} n : B}{\Gamma \vdash_{\mathrm{NF}} \lambda\, n : A \Rightarrow B}\ \textsc{Nf/}\Rightarrow\textsc{-Intro}
$$

$$
\frac{\Gamma \vdash_{\mathrm{NE}} n : A \Rightarrow B \qquad \Gamma \vdash_{\mathrm{NF}} m : A}{\Gamma \vdash_{\mathrm{NE}} \mathsf{app}\, n\, m : B}\ \textsc{Ne/}\Rightarrow\textsc{-Elim}
\qquad
\frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \qquad \Gamma, A \vdash_{\mathrm{NF}} m : B}{\Gamma \vdash_{\mathrm{NF}} \mathsf{letmap}_{\mathrm{SF}}\, n\, m : \Diamond B}\ \textsc{NF/}\Diamond\textsc{-Letmap/SF}
$$

Fig. 2. Neutral terms and Normal forms for λ_{SF}

$$
\begin{array}{l}
\mathit{reify}_{A;\Gamma} : [\![A]\!]_\Gamma \to \Gamma \vdash_{\mathrm{NF}} A \\
\mathit{reify}_{\iota;\Gamma} \quad n = \mathsf{up}\, n \\
\mathit{reify}_{\top;\Gamma} \quad u = \mathsf{unit} \\
\mathit{reify}_{A\times B;\Gamma}\ p = \mathsf{pair}\,(\mathit{reify}_{A;\Gamma}\,(\pi_1\, p))\,(\mathit{reify}_{B;\Gamma}\,(\pi_2\, p))) \\
\mathit{reify}_{A\Rightarrow B;\Gamma}\ f = \lambda\,(\mathit{reify}_{B;(\Gamma,A)}(f\ \mathit{new}_{A;\Gamma}\,(\mathit{reflect}_{A;(\Gamma,A)}\,(\mathsf{var}\,\mathsf{zero})))) \\
\mathit{reify}_{\Diamond A;\Gamma} \quad p = \mathsf{letmap}_{\mathrm{SF}}\, n\,(\mathit{reify}_{A;(\Gamma,B)}\, a) \\
\qquad \text{where } (\mathsf{single}\, n : \Gamma \lhd_{\mathrm{SF}} (\Gamma, B), a : [\![A]\!]_{\Gamma,B}) = p
\end{array}
$$

$$
\begin{array}{l}
\mathit{reflect}_{A;\Gamma} : \Gamma \vdash_{\mathrm{NE}} A \to [\![A]\!]_\Gamma \\
\mathit{reflect}_{\iota;\Gamma} \quad n = n \\
\mathit{reflect}_{\top;\Gamma} \quad n = () \\
\mathit{reflect}_{A\times B;\Gamma}\ n = (\mathit{reflect}_{A;\Gamma}\,(\mathsf{fst}\, n), \mathit{reflect}_{B;\Gamma}\,(\mathsf{snd}\, n)) \\
\mathit{reflect}_{A\Rightarrow B;\Gamma}\ n = \lambda(i : \Gamma \leq \Gamma').\, \lambda a.\, \mathit{reflect}_{B;\Gamma}(\mathsf{app}\,(\mathit{wk}_{A\Rightarrow B}\, i\, n)\,(\mathit{reify}_{A;\Gamma'}\, a)) \\
\mathit{reflect}_{\Diamond A;\Gamma} \quad n = (\mathsf{single}\, n, \mathit{reflect}_{A;(\Gamma,A)}\,(\mathsf{var}\,\mathsf{zero}))
\end{array}
$$

Fig. 3. Reification and reflection for λ_{SF}

To define *norm* we first construct a possible-world model, known as the NbE model, where contexts are worlds. By construction, we obtain an interpretation or *evaluation* of terms $[\![-]\!] : \Gamma \vdash A \to (\forall \Delta.\, [\![\Gamma]\!]_\Delta \to [\![A]\!]_\Delta)$ for this model as an instance of the interpretation of terms for an arbitrary possible-world model. We then show that this model exhibits a function $\mathit{quote} : (\forall \Delta.\, [\![\Gamma]\!]_\Delta \to [\![A]\!]_\Delta) \to \Gamma \vdash_{\mathrm{NF}} A$. With *quote*, the normalization function is simply $\mathit{norm} = \mathit{quote} \circ [\![-]\!]$.

We construct the NbE model (N, V_ι) with a frame $N = (\mathit{Ctx}, \leq, \lhd_{\mathrm{SF}})$, taking contexts for worlds, the weakening relation $\leq$ on contexts for R_i, and the modal accessibility relation $\lhd_{\mathrm{SF}}$ for R_m—both defined as follows:

$$
\mathit{base} : \cdot \leq \cdot \quad \frac{i : \Gamma \leq \Gamma'}{\mathit{drop}_A\, i : \Gamma \leq \Gamma', A} \quad \frac{i : \Gamma \leq \Gamma'}{\mathit{keep}_A\, i : \Gamma, A \leq \Gamma', A} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A}{\mathsf{single}\, n : \Gamma \lhd_{\mathrm{SF}} \Gamma, A}
$$

We use neutral terms for valuation as $V_{\iota,\Gamma} = \Gamma \vdash_{\mathrm{NE}} \iota$, and complete the construction by showing all the necessary conditions are satisfied. In particular, we can define $\mathit{incl} : \forall \Gamma, \Delta.\, \Gamma \lhd_{\mathrm{SF}} \Delta \to \Gamma \leq \Delta$ to satisfy the inclusion condition.

In the constructed NbE model, we define *quote* as:

$$
\begin{array}{l}
\mathit{quote} : (\forall \Delta.\, [\![\Gamma]\!]_\Delta \to [\![A]\!]_\Delta) \to \Gamma \vdash_{\mathrm{NF}} A \\
\mathit{quote}\ f = \mathit{reify}_{A,\Gamma}\,(f\ \mathit{idEnv}_\Gamma)
\end{array}
$$

We apply the given function f to an element $\mathit{idEnv}_\Gamma : [\![\Gamma]\!]_\Gamma$, and then *reify* the result using the auxiliary function *reify* defined in Figure 3 alongside another auxiliary function *reflect*. The functions *reify* and *reflect* are characteristic of NbE and enable the definition of *quote*. Our choice for the parameters that define frame N is key in defining these functions. In particular, our choice of $\lhd_{\mathrm{SF}}$ for R_m is key in defining the case of $\Diamond A$ in *reify* and *reflect*. The missing definition of elements such as idEnv_Γ and $\mathit{new}_{A;\Gamma} : \Gamma \leq \Gamma, A$ are given in Appendix A.4.

Towards two-dimensional semantics The simplified possible-world semantics and NbE model construction in this section can be extended to the remaining calculi by imposing additional frame conditions. This possibility can be understood by validating their characteristic axioms. Given an arbitrary possible-world frame F

- $F \models \mathrm{R}$ when R_m reflexive ($\mathit{refl}_m : \forall w.\, w\ R_m\ w$)
- $F \models \mathrm{J}$ when R_m transitive ($\mathit{trans}_m : \forall u, v, w.\, u\ R_m\ v \to v\ R_m\ w \to u\ R_m\ w$)

In the simplified approach, however, proving soundness and completeness of an equational theory for possible-world semantics (akin to categorical semantics in Proposition 1) is a far more tedious and repetitive task. These proofs require extensive and ad hoc calculus specific lemmas and offer no opportunity for reuse. The key to a modular development lies in the connection between possible-world and categorical models. We study a refined class of frames that makes this connection apparent in the upcoming Section 3 and renew our study of semantcs for the remaining calculi in Section 4 thereafter with a more efficient approach.

3 Two-Dimensional Possible-World Frames

A *two-dimensional frame* (dubbed 2-frame) is a frame $F = (W, R_i, R_m)$ (as defined earlier in Section 2.2) subjected to the following *coherence* conditions:

- $\mathit{trans}_i\ \mathit{refl}_i\ i = i$ and $\mathit{trans}_i\ i\ \mathit{refl}_i = i$
- $\mathit{trans}_i\ (\mathit{trans}_i\ i\ i')\ i'' = \mathit{trans}_i\ i\ (\mathit{trans}_i\ i'\ i'')$
- $\mathit{factor}\ \mathit{refl}_i\ m = (m, \mathit{refl}_i)$
- $\mathit{factor}\ (\mathit{trans}_i\ i_1\ i_2)\ m = (m'_2, (\mathit{trans}_i\ i'_1\ i'_2))$
  where $(i'_1, m'_1) = \mathit{factor}\ i_1\ m$ and $(i'_2, m'_2) = \mathit{factor}\ i_2\ m'_1$.

Coherence conditions impose restrictions on *how* the functions refl_i, trans_i and factor (witnessing frame conditions) compute proofs. The first two coherence conditions state that that relation R_i determines a category $\mathcal{W}_i$ whose objects are given by worlds and morphisms by proofs of R_i, with refl_i witnessing the identity morphisms and trans_i witnessing the composition of morphisms. The latter conditions on factor give us desired structure on models (see Proposition 3).

A 2-frame determines a category of covariant presheaves $\widehat{\mathcal{W}_i}$ indexed by the category $\mathcal{W}_i$. A two-dimensional model, or a *presheaf model*, $M = (F, V)$ couples a 2-frame F with a valuation V that assigns to a base type ι an object V_ι of $\widehat{\mathcal{W}_i}$.

It is well-known that categories of presheaves are cartesian-closed. Thus the category $\widehat{\mathcal{W}_i}$ is cartesian-closed and as a result a model of STLC. This suggests that we need not interpret STLC types and terms directly in $\widehat{\mathcal{W}_i}$, but can instead derive this interpretation by instantiating its categorical semantics. Similarly, showing that $\widehat{\mathcal{W}_i}$ exhibits a strong functor $\Diamond$ is sufficient to obtain a presheaf interpretation for λ_{SF}, due to Proposition 1. This section details the structures that $\widehat{\mathcal{W}_i}$ exhibits under the imposition of various conditions on 2-frames.

Proposition 3 ($\Diamond$ Functor). *The presheaf category $\widehat{\mathcal{W}_i}$ determined by a 2-frame $F = (W, R_i, R_m)$ (subject to no further conditions) exhibits an endofunctor $\Diamond$, defined for a presheaf P at some world w as $(\Diamond P)_w = \exists v.\, w\ R_m\ v \times P_v$.*

Proof. The function *factor* and the coherence conditions imposed on it ensure that $\Diamond P$ is in fact a presheaf, with *factor* giving the action of the presheaf and the coherence conditions on *factor* (e.g., $\mathit{factor}\ \mathit{refl}_i\ m = (m, \mathit{refl}_i)$) proving the presheaf laws. The functorial action of $\Diamond$ on a natural transformation $f : P \dot{\to} Q$ to yield $\Diamond f : \Diamond P \dot{\to} \Diamond Q$ is defined by applying the component of f at the world witnessing the existential quantification. The functorial laws follow immediately.

3.1 Strong Functors

A *strong* functor $F : \mathcal{C} \to \mathcal{C}$ for a cartesian category $\mathcal{C}$ is an endofunctor on $\mathcal{C}$ with a natural transformation $\theta_{P,Q} : P \times FQ \to F(P \times Q)$ natural in $\mathcal{C}$-objects P and Q such that the following diagrams stating coherence conditions commute:

$$
\begin{array}{ccc}
\mathit{1} \times FP & \xrightarrow{\theta_{\mathit{1},P}} & F(\mathit{1} \times P) \\
 & {\scriptstyle \pi_2} \searrow \quad \swarrow {\scriptstyle F\pi_2} & \\
 & FP &
\end{array}
$$

$$
\begin{array}{ccccc}
(P \times Q) \times FR & & \xrightarrow{\theta_{P \times Q,R}} & & F((P \times Q) \times R) \\
\downarrow {\scriptstyle \alpha_{P,Q,FR}} & & & & \downarrow {\scriptstyle F\alpha_{P,Q,R}} \\
P \times (Q \times FR) & \xrightarrow{id_P \times \theta_{Q,R}} & P \times F(Q \times R) & \xrightarrow{\theta_{P,Q \times R}} & F(P \times (Q \times R))
\end{array}
$$

Observe that the terminal object $\mathit{1}$, the projection morphism $\pi_2 : P \times Q \to Q$ and the associator morphism $\alpha_{P,Q,R} : (P \times Q) \times R \to P \times (Q \times R)$ (for all $\mathcal{C}$-objects P, Q, R) live in the cartesian category $\mathcal{C}$.

Proposition 4 ($\Diamond$ Strong). *The functor $\Diamond$ on $\widehat{\mathcal{W}_i}$ is a strong functor if the underlying 2-frame (W, R_i, R_m) satisfies the inclusion condition $R_m \subseteq R_i$, as witnessed by a function $\mathit{incl} : \forall w, v.\, w\ R_m\ v \to w\ R_i\ v$.*

Proof. Follows from the definition of functor $\Diamond$ in Proposition 3, and by defining the natural transformation θ using the function *incl* (following Proposition 2).

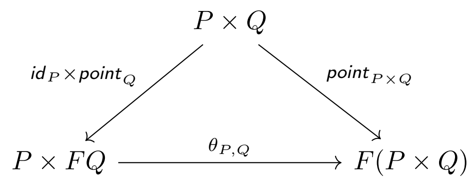
3.2 Pointed Functors

A *pointed* functor $F : \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$ is an endofunctor on $\mathcal{C}$ equipped with a natural transformation $\mathit{point} : \mathit{Id} \dot{\to} F$ from the identity functor *Id* on $\mathcal{C}$.

Proposition 5 ($\Diamond$ Pointed). *The functor $\Diamond$ on $\widehat{\mathcal{W}_i}$ is pointed if R_m is reflexive, as witnessed by a function $\mathit{refl}_m : \forall w.\, w\ R_m\ w$.*

Proof. Follows from the definition of functor $\Diamond$ in Proposition 3, and by defining the natural transformation *point* using the function refl_m.

A strong and pointed functor F is said to be *strong pointed*, when it satisfies an additional coherence condition that point is a strong natural transformation, meaning that the following diagram stating a coherence condition commutes:



Proposition 6 ($\Diamond$ Strong Pointed). *The functor $\Diamond$ on $\widehat{\mathcal{W}_i}$ is strong pointed if R_m is reflexive and satisfies inclusion condition such that $\mathsf{incl}\ \mathsf{refl}_m = \mathsf{refl}_i$.*

Proof. Propositions 4 and 5 give us that functor $\Diamond$ is strong and pointed. We use $\mathsf{incl}\ \mathsf{refl}_m = \mathsf{refl}_i$ to show that point is a strong natural transformation.

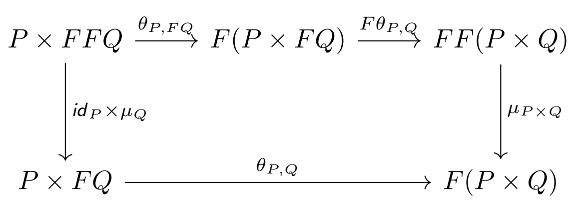
3.3 Semimonads

A *semimonad* $F : \mathcal{C} \to \mathcal{C}$, or *joinable* functor, on a category $\mathcal{C}$ is an endofunctor on $\mathcal{C}$ that forms a semigroup in the sense that it is equipped with a "multiplication" natural transformation $\mu : F^2 \dot{\to} F$ that is "associative" as $\mu_P \circ \mu_{FP} = \mu_P \circ F(\mu_P) : F^3 P \to FP$.

Proposition 7 ($\Diamond$ Semimonad). *The functor $\Diamond$ on $\widehat{\mathcal{W}_i}$ is a semimonad if R_m is associatively transitive, as witnessed by a function $\mathsf{trans}_m : \forall u, v, w.$ $u\ R_m\ v \to v\ R_m\ w \to u\ R_m\ w$ that composes some proofs m_1, m_2, m_3 of R_m such that $\mathsf{trans}_m\ (\mathsf{trans}_m\ m_1\ m_2)\ m_3 = \mathsf{trans}_m\ m_1\ (\mathsf{trans}_m\ m_2\ m_3)$.*

Proof. Follows from the definition of functor $\Diamond$ in Proposition 3, and by defining the natural transformation μ using the function trans_m.

A strong functor F that is also a semimonad is a *strong semimonad* when μ is a strong natural transformation, meaning that the following diagram stating a coherence condition commutes:



Proposition 8 ($\Diamond$ Strong Semimonad). *The functor $\Diamond$ on $\widehat{\mathcal{W}_i}$ is a strong semimonad if R_m is associatively transitive and satisfies the inclusion condition such that $\mathsf{incl}\ (\mathsf{trans}_m\ m_1\ m_2) = \mathsf{trans}_i\ (\mathsf{incl}\ m_1)\ (\mathsf{incl}\ m_2)$ for some proofs m_1, m_2.*

Proof. Propositions 4 and 7 give us that functor $\Diamond$ is strong and a semimonad. We use the given coherence condition $\mathsf{incl}\ (\mathsf{trans}_m\ m_1\ m_2) = \mathsf{trans}_i\ (\mathsf{incl}\ m_1)\ (\mathsf{incl}\ m_2)$ to show that μ is a strong natural transformation.

3.4 Monads

A *monad* $F : \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$ is a semimonad that is pointed, such that the natural transformation $\mathit{point} : \mathit{Id} \dot{\to} F$ is the left and right unit of multiplication $\mu : F^2 \dot{\to} F$ in the sense that $\mu_P \circ F\mathit{point}_P = \mathit{id}_{FP}$ and $\mu_P \circ \mathit{point}_{FP} = \mathit{id}_{FP}$ for some $\mathcal{C}$-object P.

Proposition 9 (◊ Monad). *The functor ◊ on $\widehat{\mathcal{W}_i}$ is a monad if proofs of the relation R_m form a category $\mathcal{W}_m$, with worlds for objects, refl_m witnessing the identity morphisms and trans_m witnessing the composition of morphisms.*

Proof. Propositions 5 and 7 give us that functor ◊ is a semimonad and pointed. We use the unit laws of the category $\mathcal{W}_m$ to show the unit laws of the monad.

A strong functor F that is also a monad is a *strong monad* when the natural transformations point and μ of the monad are both strong natural transformations, making F both a strong pointed functor and a strong semimonad.

Proposition 10 (◊ Strong Monad). *The functor ◊ on $\widehat{\mathcal{W}_i}$ is a strong monad if proofs of R_m form a category $\mathcal{W}_m$ and satisfies the inclusion condition such that the function incl determines an inclusion functor from $\mathcal{W}_m$ to $\mathcal{W}_i$.*

Proof. Follows immediately from Propositions 6, 8 and 9.

3.5 Constructing Presheaf Models

Figure 4 illustrates dependency in proofs of Propositions 3 to 10 and suggests opportunity for modularity in constructing presheaf models. Observe, for example, Propositions 6 and 8 both reuse Proposition 4. Models can thus be constructed for λ_{PF} and λ_{JF} using the same calculation of strength used in a model of λ_{SF}.

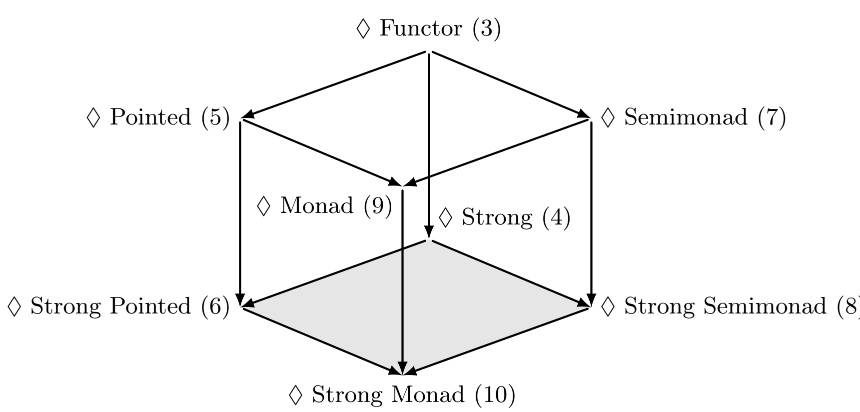


Fig. 4. Proof dependency in Propositions 3 to 10 (shade highlights strong propositions)

4 Presheaf Semantics

Two-dimensional possible-world semantics, or simply presheaf semantics, is given for a calculus by interpreting types as presheaves and terms as natural transformations. However, we need not define explicit interpretation functions $[\![-]\!]$ for types and $[\![-]\!] : \Gamma \vdash A \to [\![\Gamma]\!] \dot{\to} [\![A]\!]$ for terms as in Section 2.2. Since presheaves are objects in the category $\widehat{\mathcal{W}_i}$ with natural transformations as morphisms, we can derive the presheaf interpretation for a calculus from its categorical semantics if we show that $\widehat{\mathcal{W}_i}$ is a categorical model of the calculus.

In this section, we derive presheaf semantics for all four calculi by instantiating their respective categorical semantics. We then construct an NbE model for each calculus by instantiating its presheaf semantics and showing that the model exhibits a function $\mathit{quote} : [\![\Gamma]\!] \dot{\to} [\![A]\!] \to \Gamma \vdash_{\mathrm{NF}} A$. As in Section 2.2, quote yields a normalization function $\mathit{norm} = \mathit{quote} \circ [\![-]\!]$. We prove the correctness of normalization, i.e., $\mathit{norm}\, t \sim t$ for all t, by formulating a generic extension of Kripke logical relations suitable for all calculi. The soundness and completeness of each calculus' equational theory for presheaf semantics follows as corollaries.

4.1 Presheaf Semantics for $\lambda_{\mathbf{SF}}$

Presheaf interpretation for λ_{SF} A presheaf model of λ_{SF} is a presheaf model $\mathcal{P} = (F, V_\iota)$ where the underlying 2-frame F satisfies the inclusion condition. The interpretation of types and terms in λ_{SF}, and the soundness of the equational theory, i.e. equivalent terms have the same denotation, are given by Corollary 1.

Corollary 1 (Presheaf interpretation for $\lambda_{\mathbf{SF}}$). *For every term $\Gamma \vdash t : A$ in λ_{SF}, we have a natural transformation $[\![t]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$ in an arbitrary presheaf model $\mathcal{P}$ of λ_{SF}. Further if $\Gamma \vdash t \sim u : A$ for some u, then $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$.*

Proof. By applying Proposition 4 we get that the category $\widehat{\mathcal{W}_i}$ determined by $\mathcal{P}$ exhibits a strong functor $\Diamond$, and is thus a categorical model of λ_{SF}. The categorical interpretation of λ_{SF} in Section 2.1 (preceding Proposition 1) gives us a morphism $[\![t]\!] : [\![\Gamma]\!] \to_{\widehat{\mathcal{W}_i}} [\![A]\!]$ for every term t, which is a natural transformations $[\![t]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$. Soundness of the interpretation follows from Proposition 1.

Proposition 1 states a biimplication, but we only need the soundness half for the proof of Corollary 1. The completeness of the equational theory for presheaf semantics is a stronger statement, and we will prove it as corollary by constructing an NbE model and showing that the resulting normalization function is correct.

NbE model for λ_{SF} Recollect that we constructed the (one-dimensional) possible-world NbE model for λ_{SF} in Section 2.2 using the frame $N = (\mathit{Ctx}, \leq, \lhd_{\mathrm{SF}})$ and valuation $V_{\iota,\Gamma} = \Gamma \vdash_{\mathrm{NE}} \iota$. To construct a presheaf NbE model for λ_{SF}, we must show that N is a 2-frame that satisfies the additional coherence conditions and that V_ι is a presheaf. Both these requirements are indeed satisfiable.

$$
\begin{array}{ll}
L_{A,\Gamma} : \Gamma \vdash A \to [\![A]\!]_\Gamma \to \mathit{Type} & \\
L_{\iota;\Gamma} & t\ n = t \sim n \\
L_{\top;\Gamma} & t\ u = \top \\
L_{A\times B;\Gamma} & t\ p = L_{A;\Gamma}\,(\mathit{fst}\,t)(\pi_1\,p) \times L_{B;\Gamma}\,(\mathit{snd}\,t)(\pi_2\,p) \\
L_{A\Rightarrow B;\Gamma} & t\ f = \forall \Gamma', i : \Gamma \leq \Gamma', u, a.\, L_{A;\Gamma'}\,u\,a \to L_{B;\Gamma'}\,(\mathsf{app}\,(\mathit{wk}\,i\,t)\,u)\,(f\,i\,a) \\
L_{\Diamond A;\Gamma} & t\ p = \exists(\Delta \vdash u : A).\, t \sim \mathit{collect}\,(m,u) \times L_{A;\Gamma}\,u\,a \\
\quad \text{where } (m : \Gamma \lhd \Delta, a : [\![A]\!]_\Delta) = p &
\end{array}
$$

Fig. 5. Kripke logical relations to prove normalization

The function *quote* is also defined as before in Section 2.2 using natural transformations *reify* and *reflect* by additionally showing that the definitions in Figure 3 are natural. For the case of type $\Diamond A$, however, we opt for a more general approach that extends readily to the remaining calculi. We will use natural transformations $\mathit{collect} : \Diamond(\mathit{Nf}\ A) \dot{\to} \mathit{Nf}\,(\Diamond A)$ and $\mathit{register} : \mathit{Ne}\,(\Diamond A) \dot{\to} \Diamond(\mathit{Ne}\,A)$, where $\mathit{Nf}\ A$ and $\mathit{Ne}\,A$ are type-indexed presheaves given by normal forms and neutral terms respectively, and $\Diamond$ is the presheaf functor (from Proposition 3).

$$
\begin{array}{ll}
\mathit{reify}_{A;\Gamma} : [\![A]\!] \dot{\to} \mathit{Nf}\ A & \mathit{reflect}_{A;\Gamma} : \mathit{Ne}\,A \dot{\to} [\![A]\!] \\
\mathit{reify}_{\dots;\Gamma} \ = \dots & \mathit{reflect}_{\dots;\Gamma} \ = \dots \\
\mathit{reify}_{\Diamond A;\Gamma} = \mathit{collect} \circ \Diamond(\mathit{reify}_{A;\Gamma}) & \mathit{reflect}_{\Diamond A;\Gamma} = \Diamond(\mathit{reflect}_{A;\Gamma}) \circ \mathit{register}
\end{array}
$$

We will define *collect* and *register* seperately for each calculus, leaving definitions of *reify* and *reflect* uniform for all calculi. For λ_{SF}, they are defined as:

$$
\begin{array}{ll}
\mathit{collect}_{\mathrm{SF}} : \Diamond(\mathit{Nf}\ A) \dot{\to} \mathit{Nf}\,(\Diamond A) & \mathit{register}_{\mathrm{SF}} : \mathit{Ne}\,(\Diamond A) \dot{\to} \Diamond(\mathit{Ne}\,A) \\
\mathit{collect}_{\mathrm{SF}}\,(\mathsf{single}\,n, m) = \mathsf{letmap}_{\mathrm{SF}}\,n\,m & \mathit{register}_{\mathrm{SF}}\,n = (\mathsf{single}\,n, \mathsf{var\,zero})
\end{array}
$$

To prove correctness of the normalization function *norm*, we define a logical relation L extending the usual definition for STLC as in Figure 5. The relation L relates a term $\Gamma \vdash t : A$ to an element $t : [\![A]\!]_\Gamma$ as $L\,t\,v$ when $t \sim \mathit{reify}\,v$. Observe that we consider normal forms as terms and leave the embedding $\Gamma \vdash_{\mathrm{NF}} A \to \Gamma \vdash A$ implicit, and that we write $t \sim u$ instead of $\Gamma \vdash t \sim u : A$ for brevity.

Proposition 11 (Correctness of normalization for $\lambda_{\mathbf{SF}}$). *For all terms t in λ_{SF}, t is equivalent to its assigned normal form, i.e., $t \sim \mathit{norm}\,t$.*

Proof. We prove the "fundamental lemma" of logical relations, giving us that for any term t, we have $L\,t\,([\![t]\!]\ \mathit{idEnv})$. It follows that $t \sim \mathit{reify}\,([\![t]\!]\ \mathit{idEnv}) = \mathit{norm}\,t$.

Corollary 2 (Completeness of presheaf interpretation for $\lambda_{\mathbf{SF}}$). *For any two terms t, u in λ_{SF}, if for all presheaf models of λ_{SF} $[\![t]\!] = [\![u]\!]$ then $t \sim u$.*

Proof. In the NbE model, we know $[\![t]\!] = [\![u]\!]$ implies $\mathit{norm}\,t = \mathit{norm}\,u$. By Proposition 11, we also know $t \sim \mathit{norm}\,t$ and $u \sim \mathit{norm}\,u$, thus $t \sim u$.

Theorem 1 (Presheaf semantics for $\lambda_{\mathbf{SF}}$). *For any two terms t, u in λ_{SF}, $\Gamma \vdash t \sim u : A$ if and only if for all presheaf models of λ_{SF} $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$.*

Proof. Follows immediately from Corollaries 1 and 2.

4.2 Presheaf Semantics for $\lambda_{\mathbf{ML}}$

The calculus λ_{ML} extends STLC with constructs $\mathsf{return}_{\mathrm{ML}}$ and $\mathsf{let}_{\mathrm{ML}}$ (defined as below) and three standard equations sometimes referred to as the "monad laws" (Rule ML/$\Diamond$-β, Rule ML/$\Diamond$-η and Rule ML/$\Diamond$-ASS in Appendix A.3).

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{return}_{\mathrm{ML}}\, t : \Diamond A}\ \text{ML}/\Diamond\text{-Return} \qquad \frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : \Diamond B}{\Gamma \vdash \mathsf{let}_{\mathrm{ML}}\, t\, u : \Diamond B}\ \text{ML}/\Diamond\text{-Let}$$

A categorical model of λ_{SF} is a cartesian-closed category equipped with a strong monad $\Diamond$. Proposition 12 states the categorical semantics of λ_{ML} with the standard categorical interpretation and proof as for λ_{SF} in Proposition 1.

Proposition 12 (Categorical semantics for $\lambda_{\mathbf{ML}}$). *Given two terms t, u in λ_{ML}, $\Gamma \vdash t \sim u : A$ if and only if for all models $\mathcal{C}$ of λ_{ML} $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \to_{\mathcal{C}} [\![A]\!]$.*

A presheaf model of λ_{ML} is a presheaf model $\mathcal{P} = (F, V_\iota)$ where the underlying 2-frame $F = (W, R_i, R_m)$ satisfies the inclusion condition $R_m \subseteq R_i$ such that the witness function *incl* that determines a functor $\mathit{incl} : \mathcal{W}_m \to \mathcal{W}_i$. By applying Proposition 10, we know that F determines a strong monad $\Diamond$ on the category of presheaves $\widehat{\mathcal{W}_i}$, making $\widehat{\mathcal{W}_i}$ a categorical model of λ_{ML} as well. As a consequence, we obtain a sound presheaf interpretation for λ_{ML} as a corollary of Proposition 12, just as we did for λ_{SF} in Corollary 1.

We construct the NbE model for λ_{ML} using the 2-frame $F = (\mathit{Ctx}, \leq, \lhd_{\mathrm{ML}})$ using the following definitions of normal forms (omits those of STLC) and modal acessibility relation $\lhd_{\mathrm{ML}}$. The definition of neutral terms does not change.

$$\frac{\Gamma \vdash_{\mathrm{NF}} n : A}{\Gamma \vdash_{\mathrm{NF}} \mathsf{return}_{\mathrm{ML}}\, n : \Diamond A}\ \text{ML/NF}/\Diamond\text{-Return} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \qquad \Gamma, A \vdash_{\mathrm{NF}} m : \Diamond B}{\Gamma \vdash_{\mathrm{NF}} \mathsf{let}_{\mathrm{ML}}\, n\, m : \Diamond B}\ \text{ML/NF}/\Diamond\text{-Let}$$

$$\mathsf{nil} : \Gamma \lhd_{\mathrm{ML}} \Gamma \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \qquad m : \Gamma, A \lhd_{\mathrm{ML}} \Delta}{\mathsf{cons}\, n\, m : \Gamma \lhd_{\mathrm{ML}} \Delta}$$

Observe that relation $\lhd_{\mathrm{ML}}$ satisfies the inclusion condition and is reflexive and transitive, since we can define the functions refl_m and trans_m. These functions satistisfy all necessary coherence conditions (e.g., trans_m is associative) as well.

We can complete the definition of the normalization function and show that it is correct by defining the natural transformations *collect* and *register* as below.

$$\begin{aligned} \mathit{collect}_{\mathrm{ML}}\,(\mathsf{nil}, v) &= \mathsf{return}_{\mathrm{ML}}\, v \qquad\qquad \mathit{register}_{\mathrm{ML}}\, n = (\mathsf{cons}\, n\, \mathsf{nil}, \mathsf{var\, zero}) \\ \mathit{collect}_{\mathrm{ML}}\,(\mathsf{cons}\, n\, m, v) &= \mathsf{let}_{\mathrm{ML}}\, n\, (\mathit{collect}_{\mathrm{ML}}\,(m, v)) \end{aligned}$$

We define $\mathit{collect}_{\mathrm{ML}}$ by induction on the modal relation $\lhd_{\mathrm{ML}}$ and prove the fundamental lemma for the logical relation L in Figure 5 once again. As a result, we obtain the correctness of normalization for λ_{ML} (akin to Proposition 11) and completeness of the presheaf interpretation for λ_{ML} (akin to Corollary 2).

Theorem 2 (Presheaf semantics for $\lambda_{\mathbf{ML}}$). *For any two terms t, u in λ_{ML}, $\Gamma \vdash t \sim u : A$ if and only if for all presheaf models of λ_{ML} $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \stackrel{.}{\to} [\![A]\!]$.*

4.3 Presheaf Semantics for $\lambda_{\mathbf{PF}}$ and $\lambda_{\mathbf{JF}}$

The calculus λ_{PF} extends λ_{SF} with a construct $\mathsf{return}_{\mathrm{PF}}$, retaining $\mathsf{letmap}_{\mathrm{SF}}$ as $\mathsf{letmap}_{\mathrm{PF}}$, and an equation $\mathsf{letmap}_{\mathrm{PF}}\,(\mathsf{return}_{\mathrm{PF}}\,t)\,u \sim \mathsf{return}_{\mathrm{PF}}\,(u[t])$.

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{return}_{\mathrm{PF}}\,t : \Diamond A}\ \text{PF}/\Diamond\text{-Return}$$

A categorical model of λ_{PF} is a cartesian-closed category equipped with a strong pointed functor $\Diamond$. As before with λ_{SF} and λ_{ML}, the categorical semantics of λ_{PF} can be established. The presheaf interpretation can be derived by applying Proposition 6 for presheaf models whose frames satisfy appropriate conditions. We construct the NbE model as before with the 2-frame $(\mathit{Ctx}, \leq, \lhd_{\mathrm{PF}})$, using the following definitions of normal forms (omits those of STLC) and $\lhd_{\mathrm{PF}}$.

$$\frac{\Gamma \vdash_{\mathrm{NF}} n : A}{\Gamma \vdash_{\mathrm{NF}} \mathsf{return}_{\mathrm{PF}}\,n : \Diamond A}\ \text{PF/NF}/\Diamond\text{-Return} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \qquad \Gamma, A \vdash_{\mathrm{NF}} m : B}{\Gamma \vdash_{\mathrm{NF}} \mathsf{letmap}_{\mathrm{PF}}\,n\,m : \Diamond B}\ \text{PF/NF}/\Diamond\text{-Letmap}$$

$$\mathsf{nil} : \Gamma \lhd_{\mathrm{PF}} \Gamma \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A}{\mathsf{single}\,n : \Gamma \lhd_{\mathrm{PF}} \Gamma, A}$$

Observe that relation $\lhd_{\mathrm{PF}}$ satisfies the inclusion condition and is reflexive, but not transitive. We define natural transformations *collect* and *register* as below.

$$\begin{aligned} \mathit{collect}_{\mathrm{PF}}\,(\mathsf{nil}, m) &= \mathsf{return}_{\mathrm{PF}}\,m \\ \mathit{collect}_{\mathrm{PF}}\,(\mathsf{single}\,n, m) &= \mathsf{letmap}_{\mathrm{PF}}\,n\,m \end{aligned} \qquad \mathit{register}_{\mathrm{PF}}\,n = (\mathsf{single}\,n, \mathsf{var\ zero})$$

With the constructed NbE model, we once again prove the correctness of normalization and reproduce the presheaf semantics theorem for λ_{PF}.

Theorem 3 (Presheaf semantics for $\lambda_{\mathbf{PF}}$). *For any two terms t, u in λ_{PF}, $\Gamma \vdash t \sim u : A$ if and only if for all presheaf models of λ_{PF} $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$.*

The calculus λ_{JF} extends λ_{SF} with a construct $\mathsf{let}_{\mathrm{JF}}$, while retaining the construct $\mathsf{letmap}_{\mathrm{SF}}$ as $\mathsf{letmap}_{\mathrm{JF}}$, and three equations concerning $\mathsf{let}_{\mathrm{JF}}$ and $\mathsf{letmap}_{\mathrm{JF}}$ (see Rule JF/$\Diamond$-β_2, Rule JF/$\Diamond$-com, Rule JF/$\Diamond$-ass in Appendix A.2).

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : \Diamond B}{\Gamma \vdash \mathsf{let}_{\mathrm{JF}}\,t\,u : \Diamond B}\ \text{JF}/\Diamond\text{-Let}$$

The categorical and presheaf semantics is given following the same process for the other calculi. A categorical model of λ_{JF} is cartesian-closed category equipped with a strong semimonad $\Diamond$, and presheaf models are given by 2-frames that satisfy the frame and coherence conditions stipulated by Proposition 8.

The NbE model can be constructed using the following definitions of normal forms for λ_{JF} (omits those of STLC) and modal accessibility relation $\lhd_{\mathrm{JF}}$.

JF/NF/◇-Letmap
$$\frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \qquad \Gamma, A \vdash_{\mathrm{NF}} m : B}{\Gamma \vdash_{\mathrm{NF}} \mathsf{letmap}_{\mathrm{JF}}\, n\, m : \Diamond B}$$

JF/NF/◇-Let
$$\frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \qquad \Gamma, A \vdash_{\mathrm{NF}} m : \Diamond B}{\Gamma \vdash_{\mathrm{NF}} \mathsf{let}_{\mathrm{JF}}\, n\, m : \Diamond B}$$

$$\frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A}{\mathsf{single}\, n : \Gamma \lhd_{\mathrm{JF}} \Gamma, A} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \qquad m : \Gamma, A \lhd_{\mathrm{JF}} \Delta}{\mathsf{cons}\, n\, m : \Gamma \lhd_{\mathrm{JF}} \Delta}$$

Observe that relation $\lhd_{\mathrm{PF}}$ satisfies the inclusion condition and is transitive, but not reflexive. We define natural transformations *collect* and *register* as below.

$$\begin{aligned} \mathit{collect}_{\mathrm{JF}}\,(\mathsf{single}\, n, v) &= \mathsf{letmap}_{\mathrm{JF}}\, n\, v \qquad \mathit{register}_{\mathrm{JF}}\, n = (\mathsf{single}\, n, \mathsf{var\ zero}) \\ \mathit{collect}_{\mathrm{JF}}\,(\mathsf{cons}\, n\, m, v) &= \mathsf{let}_{\mathrm{JF}}\, n\, (\mathit{collect}_{\mathrm{JF}}\,(m, v)) \end{aligned}$$

The presheaf semantics theorem follows the same argument as before.

Theorem 4 (Presheaf semantics for $\lambda_{\mathbf{JF}}$). *For any two terms t, u in λ_{JF}, $\Gamma \vdash t \sim u : A$ if and only if for all presheaf models of λ_{JF} $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$.*

5 Related and Further Work

Our interpretation of strong functors as an intuitionistic possibility modality is based on the observation that a normal form $\Gamma \vdash_{\mathrm{NF}} \Diamond A$ in all our calculi consists of a normal form of type A in *some* extended context $\Gamma, A_1, A_2 ... A_n$. We capture this context extension using a proof-relevant instantiation of the relation R_m.

The concept of intuitionistic possibility as a fixed modality with a corresponding set of frame conditions does not appear to have reached a consensus. Our interpretation and the frame condition ${R_i}^{-1}; R_m \subseteq R_m; {R_i}^{-1}$ can be found in Plotkin and Stirling [28, Section 1] and Simpson [29, Chapter 3.3 ("F1")]. Simpson [29] gives a comprehensive summary of a myriad of different frame conditions that have been used to study intuitionistic necessity and possibility. Discussions concerning frame conditions and other proof theoretic considerations in intuitionistic modal logic can as well be found in later work [4,27,16,22,12].

Fairtlough and Mendler [13] devise *propositional lax logic* (PLL) as an extension of intuitionistic propositional logic with a "curious modality ◯" that "has a flavour of both possibility and of necessity". Motivated by constraint solving in hardware verification, they give the following possible-world interpretation:

$$[\![\bigcirc A]\!]_w = \forall w'.\, w\ R_i\ w' \to \exists v.\, w'\ R_m\ v \times [\![A]\!]_v$$

A PLL formula $\bigcirc A$ is to be understood as a formula A that is subject to *some* set of verification constraints. PLL exhibits the axioms $\mathrm{R} : A \Rightarrow \bigcirc A$, $\mathrm{J} : \bigcirc\bigcirc A \Rightarrow \bigcirc A$ and $\mathrm{S}' : \bigcirc A \times \bigcirc B \Rightarrow \bigcirc(A \times B)$. They formulate Hilbert and Gentzen style proof systems and give possible-world semantics for PLL by requiring the relation R_m to be reflexive, transitive and satisfy $R_m \subseteq R_i$. They also show the deductive soundness and completeness of their proof systems for possible-world semantics.

Benton, Bierman and de Paiva [6] independently formulate a logic equivalent to PLL called *CL-logic* from a logical reconstruction of λ_{ML}. They devise categorical and possible-world semantics for CL-logic and leave open the question of the connection in between them. Their possible-world semantics requires the relation R_m to be hereditary, meaning if $[\![A]\!]_w$ and $w\ R_m\ v$ then $[\![A]\!]_v$, and they show deductive completeness using a Henkin-model construction.

The logical equivalence between PLL and CL-logic carries over to presheaves. The modality $\bigcirc$ determines a functor $\bigcirc$ on $\widehat{\mathcal{W}_i}$ that is naturally ismorphic to $\Diamond$.

Proposition 13. *The presheaf functors $\Diamond$ and $\bigcirc$ determined by an arbitrary 2-frame (subject to no further conditions) are naturally isomorphic.*

Alechina at al. [4] study a connection between categorical and possible-world models of PLL/CL-logic. They show that a PLL-*modal algebra* determines a possible-world model of PLL [4, Theorem 4] via the Stone representation, and observe that a modal algebra is a thin categorical model (i.e., whose morphisms are given by the partial-order relation of the algebra). This connection, while illuminating, is insufficient for our purposes of constructing NbE models. Our approach is to instead show that possible-world frames determine presheaf models, which are themselves categorical models. In specific, we follow Kavvos [17] in observing that a refined class of so-called two-dimensional possible-world models *are* presheaf models. We do not, however, study the connection to profunctors.

The connection of our work to classical modal logic can be observed (in a classical meta-theory) by instantiating the relation R_i to the identity relation. It can be shown that the remaining conditions on frames are necessary and sufficient for validating the classical modal axioms S, R and J. The study of necessary and sufficient frame conditions, known as *frame correspondence*, appears to be tricky in the intuitionistic setting. Plotkin and Stirling [28] prove a correspondence theorem that gives us that the corresponding frame conditions for axioms R and J are reflexivity of $R_m; {R_i}^{-1}$ and ${R_m}^2 \subseteq R_m; {R_i}^{-1}$ respectively. We have not studied frame correspondence in this article, but leave it as a matter for future work that can demystify the zoo of frame conditions in literature. A categorical perspective like that of Kavvos [17] is likely to be beneficial here.

Coquand [10,11] mechanized the completeness of STLC with explicit substitutions for possible-world semantics by constructing an NbE model in the proof assistant ALF [23]. Possible-world semantics for STLC does not involve the modal accessibility relation R_m. The role of the modal accessibility relation in NbE is a relatively recent consideration, and has been studied for Fitch-style modal calculi with necessity modalities [31]. NbE for variants and extensions of λ_{ML} has been proved correct and mechanized on multiple occasions [14,3,30], and we have merely reconstructed NbE for λ_{ML} systematically. Our algorithm is somewhat extensible: we can extend λ_{ML} freely with arbitrary uninterpreted monadic primitives by augmenting the definition of relation $\lhd_{\mathrm{ML}}$. The addition of sum types, however, is a delicate matter [5,15,2,32] that requires further work.

The outcome of this article is not complete possible-world semantics or NbE for *all* strong functor calculi, but rather the identification of minimum structure that all their possible-world and NbE models must inevitably possess.

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A Term Calculi

A.1 The calculus $\lambda_{\mathbf{PF}}$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{return}_{\mathrm{PF}}\, t : \Diamond A}\ \text{PF}/\Diamond\text{-Return} \qquad \frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : B}{\Gamma \vdash \mathsf{letmap}_{\mathrm{PF}}\, t\, u : \Diamond B}\ \text{PF}/\Diamond\text{-Letmap}$$

$$\frac{\Gamma \vdash t : \Diamond A}{\Gamma \vdash t \sim \mathsf{letmap}_{\mathrm{PF}}\, t\, (\mathsf{var\ zero}) : \Diamond A}\ \text{PF}/\Diamond\text{-}\eta$$

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : B \qquad \Gamma, B \vdash u' : C}{\Gamma \vdash \mathsf{letmap}_{\mathrm{PF}}\, (\mathsf{letmap}_{\mathrm{PF}}\, t\, u)\, u' \sim \mathsf{letmap}_{\mathrm{PF}}\, t\, (u'[u]) : \Diamond C}\ \text{PF}/\Diamond\text{-}\beta_1$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma, A \vdash u : B}{\Gamma \vdash \mathsf{letmap}_{\mathrm{PF}}\, (\mathsf{return}_{\mathrm{PF}}\, t)\, u \sim \mathsf{return}_{\mathrm{PF}}\, (u[t]) : \Diamond B}\ \text{PF}/\Diamond\text{-}\beta_2$$

Fig. 6. Well-typed terms and equational theory for λ_{PF} (omitting those of STLC)

Proposition 14 (Categorical semantics for $\lambda_{\mathbf{PF}}$). *Given two terms t, u in λ_{PF}, $\Gamma \vdash t \sim u : A$ if and only if for all models $\mathcal{C}$ of λ_{PF} $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \to_{\mathcal{C}} [\![A]\!]$.*

Corollary 3 (Presheaf interpretation for $\lambda_{\mathbf{PF}}$). *For every term $\Gamma \vdash t : A$ in λ_{PF}, we have a natural transformation $[\![t]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$ in an arbitrary presheaf model $\mathcal{P}$ of λ_{PF}. Further if $\Gamma \vdash t \sim u : A$ for some u, then $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$.*

Proposition 15 (Correctness of normalization for $\lambda_{\mathbf{PF}}$). *For all terms t in λ_{PF}, t is equivalent to its assigned normal form, i.e., $t \sim \mathsf{norm}\, t$.*

Corollary 4 (Completeness of presheaf interpretation for $\lambda_{\mathbf{PF}}$). *For any two terms t, u in λ_{PF}, if for all presheaf models of λ_{PF} $[\![t]\!] = [\![u]\!]$ then $t \sim u$.*

A.2 The calculus $\lambda_{\mathbf{JF}}$

JF/$\Diamond$-Letmap

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : B}{\Gamma \vdash \mathsf{letmap}_{\mathrm{JF}}\, t\, u : \Diamond B}$$

JF/$\Diamond$-Let

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : \Diamond B}{\Gamma \vdash \mathsf{let}_{\mathrm{JF}}\, t\, u : \Diamond B}$$

JF/$\Diamond$-η

$$\frac{\Gamma \vdash t : \Diamond A}{\Gamma \vdash t \sim \mathsf{letmap}_{\mathrm{JF}}\, t\, (\mathsf{var\ zero}) : \Diamond A}$$

JF/$\Diamond$-β_1

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : B \qquad \Gamma, B \vdash u' : C}{\Gamma \vdash \mathsf{letmap}_{\mathrm{JF}}\, (\mathsf{letmap}_{\mathrm{JF}}\, t\, u)\, u' \sim \mathsf{letmap}_{\mathrm{JF}}\, t\, (u'[u]) : \Diamond C}$$

JF/$\Diamond$-β_2

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : B \qquad \Gamma, B \vdash u' : \Diamond C}{\Gamma \vdash \mathsf{let}_{\mathrm{JF}}\, (\mathsf{letmap}_{\mathrm{JF}}\, t\, u)\, u' \sim \mathsf{let}_{\mathrm{JF}}\, t\, (u'[u]) : \Diamond C}$$

JF/$\Diamond$-com

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : \Diamond B \qquad \Gamma, B \vdash u' : C}{\Gamma \vdash \mathsf{letmap}_{\mathrm{JF}}\, (\mathsf{let}_{\mathrm{JF}}\, t\, u)\, u' \sim \mathsf{let}_{\mathrm{JF}}\, t\, (\mathsf{letmap}_{\mathrm{JF}}\, u\, (\mathit{wk}\, u')) : \Diamond C}$$

JF/$\Diamond$-ass

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : \Diamond B \qquad \Gamma, B \vdash u' : \Diamond C}{\Gamma \vdash \mathsf{let}_{\mathrm{JF}}\, (\mathsf{let}_{\mathrm{JF}}\, t\, u)\, u' \sim \mathsf{let}_{\mathrm{JF}}\, t\, (\mathsf{let}_{\mathrm{JF}}\, u\, (\mathit{wk}\, u')) : \Diamond C}$$

Fig. 7. Well-typed terms and equational theory for λ_{JF} (omitting those of STLC)

Proposition 16 (Categorical semantics for $\lambda_{\mathbf{JF}}$). *Given two terms t, u in λ_{JF}, $\Gamma \vdash t \sim u : A$ if and only if for all models $\mathcal{C}$ of λ_{JF} $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \to_{\mathcal{C}} [\![A]\!]$.*

Corollary 5 (Presheaf interpretation for $\lambda_{\mathbf{JF}}$). *For every term $\Gamma \vdash t : A$ in λ_{JF}, we have a natural transformation $[\![t]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$ in an arbitrary presheaf model $\mathcal{P}$ of λ_{JF}. Further if $\Gamma \vdash t \sim u : A$ for some u, then $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$.*

Proposition 17 (Correctness of normalization for $\lambda_{\mathbf{JF}}$). *For all terms t in λ_{JF}, t is equivalent to its assigned normal form, i.e., $t \sim \mathsf{norm}\, t$.*

Corollary 6 (Completeness of presheaf interpretation for $\lambda_{\mathbf{JF}}$). *For any two terms t, u in λ_{JF}, if for all presheaf models of λ_{JF} $[\![t]\!] = [\![u]\!]$ then $t \sim u$.*

A.3 The calculus $\lambda_{\mathbf{ML}}$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{return}_{\mathrm{ML}}\, t : \Diamond A}\ \text{ML}/\Diamond\text{-Return} \qquad \frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : \Diamond B}{\Gamma \vdash \mathsf{let}_{\mathrm{ML}}\, t\, u : \Diamond B}\ \text{ML}/\Diamond\text{-Let}$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma, A \vdash u : \Diamond B}{\Gamma \vdash \mathsf{let}_{\mathrm{ML}}\, (\mathsf{return}_{\mathrm{ML}}\, t)\, u \sim u[t] : \Diamond B}\ \text{ML}/\Diamond\text{-}\beta \qquad \frac{\Gamma \vdash t : \Diamond A}{\Gamma \vdash t \sim \mathsf{let}_{\mathrm{ML}}\, t\, (\mathsf{return}_{\mathrm{ML}}\, (\mathsf{var\,zero})) : \Diamond A}\ \text{ML}/\Diamond\text{-}\eta$$

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : \Diamond B \qquad \Gamma, B \vdash u' : \Diamond C}{\Gamma \vdash \mathsf{let}_{\mathrm{ML}}\, (\mathsf{let}_{\mathrm{ML}}\, t\, u)\, u' \sim \mathsf{let}_{\mathrm{ML}}\, t\, (\mathsf{let}_{\mathrm{ML}}\, u\, (\mathit{wk}\, u')) : \Diamond C}\ \text{ML}/\Diamond\text{-ass}$$

Fig. 8. Well-typed terms and equational theory for λ_{ML} (omitting those of STLC)

Corollary 7 (Presheaf interpretation for $\lambda_{\mathbf{ML}}$). *For every term $\Gamma \vdash t : A$ in λ_{ML}, we have a natural transformation $[\![t]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$ in an arbitrary presheaf model $\mathcal{P}$ of λ_{ML}. Further if $\Gamma \vdash t \sim u : A$ for some u, then $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$.*

Proposition 18 (Correctness of normalization for $\lambda_{\mathbf{ML}}$). *For all terms t in λ_{ML}, t is equivalent to its assigned normal form, i.e., $t \sim \mathsf{norm}\, t$.*

Corollary 8 (Completeness of presheaf interpretation for $\lambda_{\mathbf{ML}}$). *For any two terms t, u in λ_{ML}, if for all presheaf models of λ_{ML} $[\![t]\!] = [\![u]\!]$ then $t \sim u$.*

A.4 Auxiliary definitions

– The relation $\leq$ is reflexive and transitive, as witnessed by functions:

$$
\begin{array}{ll}
\mathit{refl}_{\leq\Gamma} : \Gamma \leq \Gamma & \mathit{trans}_{\leq} : \Gamma \leq \Gamma' \to \Gamma' \leq \Gamma'' \to \Gamma \leq \Gamma'' \\
\mathit{refl}_{\leq\cdot} \quad = \mathit{base} & \mathit{trans}_{\leq}\ i \quad \mathit{base} \quad = i \\
\mathit{refl}_{\leq\Gamma,A} = \mathit{keep}_A\ \mathit{refl}_{\leq\Gamma} & \mathit{trans}_{\leq}\ i \quad (\mathit{drop}\, i') = \mathit{drop}\,(\mathit{trans}_{\leq}\, i\, i') \\
& \mathit{trans}_{\leq}\ (\mathit{drop}\, i)\ (\mathit{keep}\, i') = \mathit{drop}\,(\mathit{trans}_{\leq}\, i\, i') \\
& \mathit{trans}_{\leq}\ (\mathit{keep}\, i)\ (\mathit{keep}\, i') = \mathit{keep}\,(\mathit{trans}_{\leq}\, i\, i')
\end{array}
$$

– The proof element *new* is defined as:

$$
\begin{array}{l}
\mathit{new}_{A;\Gamma} : \Gamma \leq \Gamma, A \\
\mathit{new}_{A;\Gamma} = \mathit{drop}_A\ \mathit{refl}_{\leq\Gamma}
\end{array}
$$

– The function *factor* for the NbE model of λ_{SF} is defined as:

$$
\begin{array}{l}
\mathit{factor} : \forall \Gamma, \Gamma', \Delta.\, \Gamma \leq \Gamma' \to \Gamma \lhd_{\mathrm{SF}} \Delta \to \exists \Delta'.\, (\Gamma' \lhd_{\mathrm{SF}} \Delta' \times \Delta \leq \Delta') \\
\mathit{factor}\ i\ (\mathsf{single}\, n) = (\mathsf{single}\,(\mathit{wk}\, i\, n), \mathit{keep}\, i)
\end{array}
$$

– The function *incl* for the NbE model of λ_{SF} is defined as:

$$
\begin{array}{l}
\mathit{incl} : \forall \Gamma, \Delta.\, \Gamma \lhd_{\mathrm{SF}} \Delta \to \Gamma \leq \Delta \\
\mathit{incl}\ (\mathsf{single}\,(n : \Gamma \vdash_{\mathrm{NE}} \Diamond A)) = \mathit{new}_{A;\Gamma}
\end{array}
$$

– The element *idEnv* is defined in the NbE model of λ_{SF} (and others) as:

$$
\begin{array}{l}
\mathit{idEnv}_{\Gamma} : [\![\Gamma]\!]_{\Gamma} \\
\mathit{idEnv}_{\cdot} \quad = () \\
\mathit{idEnv}_{\Gamma,A} = (\mathit{wk}_{\Gamma}\ \mathit{new}_{A;\Gamma}\ \mathit{idEnv}_{\Gamma}, \mathit{reflect}_{A;\Gamma}\ (\mathsf{var\,zero}))
\end{array}
$$