Fitch-Style Applicative Functors (Extended Abstract)

Nachiappan Valliappan
Chalmers University of Technology
Göteborg, Sweden

Fabian Ruch
Unaffiliated
Göteborg, Sweden

Abstract

Type systems with modalities have found a wide range of applications in programming languages to capture and specify properties of a program in its type. Since different applications may demand different modal operations, we desire a uniform approach to designing and studying modal lambda calculi. Fitch-style modal calculi simplify formulating calculi that incorporate different modal operations using the characteristic 🔒 operator on contexts. The presence of 🔒, however, complicates reasoning with the syntax of the calculus, and demands tedious and seemingly ad hoc treatment to prove meta-theoretic properties such as normalization.

It has been shown that normalization can be achieved for some Fitch-style calculi by constructing Normalization by Evaluation (NbE) models as instances of their possible-world semantics, thus bypassing reasoning about the intricate syntax of Fitch-style calculi. In this article, we pursue the extension of this result to a Fitch-style calculus for *applicative functors*. We also discuss the applicability of this calculus with some concrete applicative functors.

1 Introduction

Type systems with modalities enable us to capture and specify properties of a program explicitly in its type. A modality can be understood as a unary type former with some operations. The *necessity* modality, which originates from modal logic, is a type former $\Box$ accompanied by the *necessitation rule* (if $\cdot \vdash A$ then $\Gamma \vdash \Box A$) and the *K axiom* ($\Box(A \Rightarrow B) \Rightarrow \Box A \Rightarrow \Box B$). Type systems with necessity modalities have found several applications in programming languages including modelling purity in an impure functional language ($\Box A$ as a pure value of type A) [4], confidentiality in information-flow control ($\Box A$ as a secret of type A) [15], and binding-time separation in partial evaluation and staged computation ($\Box A$ as code of type A) [6, 7].

Different applications may demand the addition of different modal operations, and the inclusion of each operation results in a different type system. From the perspective of modal logic, these operations correspond to modal axioms, and the inclusion of each axiom results in a different modal logic. For example, by adding the modal axioms T ($\Box A \Rightarrow A$) and 4 ($\Box A \Rightarrow \Box\Box A$) to the basic modal logic IK with the necessitation rule and axiom K, we obtain the richer logics IT (adding axiom T), IK4 (adding axiom 4), and IS4 (adding both T and 4). Formulating type systems that correspond to different modal logics with well-behaved semantic and computational properties is an active area of study, and the topic of this article is a type system which includes the following axiom:

$$\mathrm{R} : A \Rightarrow \Box A$$

Fitch-style modal lambda calculi [3, 5, 13] feature necessity modalities in a typed lambda calculus by extending the typing context with a delimiting "lock" operator (denoted by 🔒). The characteristic 🔒 operator simplifies formulating calculi that incorporate different modal axioms and these calculi have elegant semantic and computational properties. In this article, we study the Fitch-style calculus λ_{IR} presented by Clouston [5], which corresponds to the logic IR obtained by adding axiom R to IK.

The rules of λ_{IR} are summarized in Figure 1, where the omitted rules for λ-abstraction and function application are formulated in the usual way. This calculus exhibits the interface of *applicative functors* [14] in functional programming for the type former $\Box$, where the derivation of axiom R gives us $\mathsf{pure} : A \Rightarrow \Box A$ and that of axiom K gives us $(\circledast) : \Box(A \Rightarrow B) \Rightarrow \Box A \Rightarrow \Box B$.

$$\text{Ty} \quad A ::= \ldots \mid \Box A \qquad \text{Ctx} \quad \Gamma ::= \cdot \mid \Gamma, x : A \mid \Gamma, \text{🔒}$$

$$\textsc{Var}\ \frac{}{\Gamma, x : A, \Gamma' \vdash x : A} \qquad \Box\text{-}\textsc{Intro}\ \frac{\Gamma, \text{🔒} \vdash t : A}{\Gamma \vdash \mathsf{box}\, t : \Box A}$$

$$\Box\text{-}\textsc{Elim}\ \frac{\Gamma \vdash t : \Box A}{\Gamma, \text{🔒}, \Gamma' \vdash \mathsf{unbox}_{\lambda_{\mathrm{IR}}}\, t : A}$$

Figure 1. Typing rules for λ_{IR} (omitting λ-abstraction and application)

The presence of 🔒, however, complicates reasoning with the syntax of the calculus, and demands tedious and seemingly ad hoc treatment to prove meta-theoretic properties such as normalization. It has been shown that normalization can be achieved for some Fitch-style calculi by constructing Normalization by Evaluation (NbE) [2] models as instances of their possible-world semantics [18], thus bypassing reasoning about the intricate syntax of Fitch-style calculi. In this article, we pursue the extension of this result to the Fitch-style calculus λ_{IR}. In particular, we identify the *frame* condition (defined in Section 2) that defines the possible-world models of λ_{IR} and suggest a normalization algorithm for it by constructing an NbE model instance.

The main challenge encountered in giving possible-world semantics for λ_{IR} is that it varies from other Fitch-style calculi in both the VAR and the $\Box$-ELIM rules. This is unlike the Fitch-style calculi considered in previous work by Valliappan, Ruch, and Tomé Cortiñas [18] which only vary in the $\Box$-elimination rule. Moreover, their work leverages the well-understood frame conditions of axioms T and 4 in *classical* modal logic. Axiom R does not, however, have any interesting possible-world semantics in classical modal logic, since the models that satisfy the corresponding frame condition are degenerate. We elaborate further on this gap in Section 5.

In the next section, we provide an overview of possible-world semantics of Fitch-style calculi, present the frame condition that enables the interpretation of axiom R, and illustrate the construction of an NbE model instance for λ_{IR}. In Section 3, we define the calculus λ_{IR} in further detail and discuss its interpretation in a possible-world model. We elaborate on the applicative functors that can be represented using λ_{IR} and give examples of three such applicative functors in Section 4.

2 Possible-World Semantics and NbE

The possible-world semantics enable a uniform treatment of various Fitch-style calculi by isolating their differences to the specific model parameter known as a *frame*. The types and contexts of all Fitch-style calculi are interpreted alike in a possible-world model, and the interpretation of a specific calculus and the construction of an NbE model instance is enabled by imposing further conditions on frames. In this section, we first discuss the common interpretation of types and contexts of Fitch-style calculi, and then present the frame condition for interpreting λ_{IR} and constructing its NbE model.

Possible-World Models. A possible-world model is given by a frame F and a *valuation* V_ι. A frame F is a triple that consists of a type W of *worlds* and two binary *accessibility* relations R_i (for "intuitionistic") and R_m (for "modal") on worlds that are required to satisfy the following frame conditions (subject to certain coherence conditions [18]):

- R_i is *reflexive* and *transitive*
- $R_m \,;\, R_i \subseteq R_i \,;\, R_m$.

For worlds $w, w', v : W$, we may think of $w \; R_i \; w'$ as an increase in knowledge from w to w' and $w \; R_m \; v$ as a possible passage of time from w to v. The latter condition states that if $w \; R_m \; v$ and $v \; R_i \; v'$ then there exists some world w' such that $w \; R_i \; w'$ and $w' \; R_m \; v'$. A valuation V_ι is a family of types $V_{\iota,w}$ indexed by a world w, along with functions wk_A : $V_{\iota,w} \to V_{\iota,w'}$ whenever $w \; R_i \; w'$.

Interpreting Fitch-Style Calculi. Given a possible-world model, we interpret (object) types A in a Fitch-style calculus as families of (meta) types $[\![A]\!]_w$ indexed by worlds $w : W$ as below:

$$\begin{aligned}
[\![\iota]\!]_w &= V_{\iota,w} \\
[\![A \Rightarrow B]\!]_w &= \forall w'.\, w \; R_i \; w' \to [\![A]\!]_{w'} \to [\![B]\!]_{w'} \\
[\![\Box A]\!]_w &= \forall w'.\, w \; R_i \; w' \to \forall v.\, w' \; R_m \; v \to [\![A]\!]_v
\end{aligned}$$

The base type ι is interpreted using the valuation V_ι, and function types $A \Rightarrow B$ at world w are interpreted as families of functions $[\![A]\!]_{w'} \to [\![B]\!]_{w'}$ for $w \; R_i \; w'$. The interpretation of types $\Box A$ can be understood as a statement about the future: $\Box A$ is true at a world w if A is necessarily true in any possible future world v of world w' for $w \; R_i \; w'$.

We interpret contexts Γ in a Fitch-style calculus also as families of types $[\![A]\!]_w$ indexed by worlds $w : W$ as below:

$$\begin{aligned}
[\![\cdot]\!]_w &= \top \\
[\![\Gamma, A]\!]_w &= [\![\Gamma]\!]_w \times [\![A]\!]_w \\
[\![\Gamma, 🔒]\!]_w &= \textstyle\sum_u u \; R_m \; w \times [\![\Gamma]\!]_u
\end{aligned}$$

The empty context and the context extension are interpreted as usual by the Cartesian product of families. Dual to the interpretation of types $\Box A$, the interpretation of contexts Γ, 🔒 can be understood as a statement about the past: Γ, 🔒 is true at a world w if Γ is true at some past world u for which w is a possibility.

The frame conditions imposed on a possible-world model enable the interpretation of terms in a Fitch-calculus, defined by a function $[\![-]\!] : \Gamma \vdash A \to (\forall w.\, [\![\Gamma]\!]_w \to [\![A]\!]_w)$. In particular, these conditions are required to prove the *monotonicity* lemma which is necessary for the interpretation to be sound. For types A, this lemma states that we have $\text{wk}_A : [\![A]\!]_w \to [\![A]\!]_{w'}$ whenever $w \; R_i \; w'$—and similarly for contexts Γ.

The Fitch-style calculus λ_{IK}, corresponding to the basic modal logic IK, can be interpreted in an arbitrary possible-world model and an NbE model can also be constructed as an instance. To achieve the same for Fitch-style calculi corresponding to richer logics, we must impose further frame conditions on the definition of a possible-world model. For example, to interpret λ_{IT}, corresponding to the logic IT featuring axiom T : $\Box A \Rightarrow A$, in a possible-world model, we further require R_m to be a reflexive relation. With this requirement, we have that $[\![\Box A]\!]_w$ implies $[\![A]\!]_w$ for an arbitrary world w, since both R_i and R_m are reflexive now, thus validating axiom T in the model.

Frame Condition for λ_{IR}. It is also possible to validate axiom R by requiring a frame condition on R_m, which states that worlds be isolated with respect to the R_m relation, i.e. $w \; R_m \; v$ implies $w = v$ for all $w, v : W$. This condition, however, severely restricts the models which can be constructed as instances, and, in particular, exempts the construction of an NbE model for λ_{IR}. In Section 5, we discuss why an attempt to impose a condition on R_m alone is likely to be a futile one. But do not despair! Fortunately, we can instead validate the axiom with the following frame condition (subject to a coherence condition):

$$R_m \subseteq R_i$$

With this condition, we now have that $[\![A]\!]_w$ implies $[\![\Box A]\!]_w$ for an arbitrary world w. This is because, given $[\![A]\!]_w$, $w\ R_i\ w'$ and $w'\ R_m\ v$, we have $w\ R_i\ v$ using the frame condition $R_m \subseteq R_i$ and transitivity of R_i, from which we obtain the desired $[\![A]\!]_v$ by applying the monotonicity lemma.

NbE Model for λ_{IR}. The construction of an NbE model is performed alike for all Fitch-style calculi, including λ_{IR}. A possible-world model is an NbE model for a Fitch-style calculus when it is equipped with a function quote : $(\forall w.\, [\![\Gamma]\!]_w \to [\![A]\!]_w) \to \Gamma \vdash_{\mathrm{NF}} A$ that inverts the denotation $(\forall w.\, [\![\Gamma]\!]_w \to [\![A]\!]_w)$ of a term to a derivation $\Gamma \vdash_{\mathrm{NF}} A$ in normal form. The normal forms of λ_{IR} are defined below by extending the usual normal forms in the non-modal fragment.

$$\textsc{Nf}/\Box\textsc{-Intro}\ \frac{\Gamma, \text{🔒} \vdash_{\mathrm{NF}} t : A}{\Gamma \vdash_{\mathrm{NF}} \mathsf{box}\, t : \Box A} \qquad \textsc{Ne}/\Box\textsc{-Elim}\ \frac{\Gamma \vdash_{\mathrm{NE}} t : \Box A}{\Gamma, \text{🔒}, \Gamma' \vdash_{\mathrm{NE}} \mathsf{unbox}_{\lambda_{\mathrm{IK}}}\, t : A}$$

The judgement $\Gamma \vdash_{\mathrm{NE}} A$ denotes a special case of normal forms known as *neutral elements* and is defined mutually with normal forms as usual.

To construct an NbE model, we instantiate the parameters that define possible-world models as follows: we pick contexts for W, *renamings* $\Gamma' \vdash_{\mathrm{R}} \Gamma$ (defined in the next section) for $\Gamma\ R_i\ \Gamma'$, and neutral derivations $\Gamma \vdash_{\mathrm{NE}} \iota$ as the valuation $V_{\iota,\Gamma}$. We pick the modal accessibility relation $\Delta\ R_m\ \Gamma$ by observing the relationship between the contexts in the premise and conclusion of the $\Box$-Elim rule, pictured as:

$$\Box\textsc{-Elim}\ \frac{\Gamma \vdash t : \Box A}{\Delta \vdash \mathsf{unbox}\, t : A}\ (\Gamma \lhd \Delta)$$

The relation $\Gamma \lhd \Delta$ for λ_{IR} is defined as $\exists \Gamma'.\, \Delta = \Gamma, \text{🔒}, \Gamma'$.

The function quote is implemented as for other Fitch-style calculi using *reification*, which is defined as a family of functions $\mathrm{reify}_A : \forall \Gamma.\, [\![A]\!]_\Gamma \to \Gamma \vdash_{\mathrm{NF}} A$ indexed by a type A. To reify a value of $[\![\Box A]\!]_\Gamma$, we crucially leverage the fact that $\Gamma \lhd \Gamma, \text{🔒}$. Observe that $[\![\Box A]\!]_\Gamma = \forall \Gamma'.\, \Gamma' \vdash_{\mathrm{R}} \Gamma \to \forall \Delta.\, \Gamma' \lhd \Delta \to [\![A]\!]_\Delta$ by definition of $[\![-]\!]$ and the above instantiations. By picking Γ for Γ' and $\Gamma, \text{🔒}$ for Δ, we get $[\![A]\!]_{\Gamma,\text{🔒}}$ since $\vdash_{\mathrm{R}}$ is reflexive and we know $\Gamma \lhd \Gamma, \text{🔒}$. By recursively reifying $[\![A]\!]_{\Gamma,\text{🔒}}$, we get a normal form $\Gamma, \text{🔒} \vdash_{\mathrm{NF}} n : A$, which can be used to construct the desired normal form $\Gamma \vdash_{\mathrm{NF}} \mathsf{box}\, n : \Box A$ using the rule Nf/$\Box$-Intro.

3 The Calculus λ_{IR}

Syntax, Substitutions and Equational Theory. The syntax of λ_{IR} is given as in Figure 1, where we may choose to replace Rule Var with an equivalent rule that is defined inductively. To capture the role of the modal accessibility relation explicitly in the syntax of λ_{IR}, we may also replace the elimination rule $\Box$-Elim with the following rule and the relation $\lhd_{\lambda_{\mathrm{IR}}}$ defined inductively in Figure 2.

$$\Box\textsc{-Elim}\ \frac{\Delta \vdash t : \Box A \qquad e : \Delta \lhd_{\lambda_{\mathrm{IR}}} \Gamma}{\Gamma \vdash \mathsf{unbox}_{\lambda_{\mathrm{IR}}}\, t\, e : A}$$

$$\frac{}{\mathsf{nil} : \Gamma \lhd_{\lambda_{\mathrm{IR}}} \Gamma, \text{🔒}} \qquad \frac{e : \Delta \lhd_{\lambda_{\mathrm{IR}}} \Gamma}{\mathsf{var}\, e : \Delta \lhd_{\lambda_{\mathrm{IR}}} \Gamma, A}$$

$$\frac{e : \Delta \lhd_{\lambda_{\mathrm{IR}}} \Gamma}{\mathsf{lock}\, e : \Delta \lhd_{\lambda_{\mathrm{IR}}} \Gamma, \text{🔒}}$$

Figure 2. Modal accessibility relation on contexts (λ_{IR})

We define substitutions inductively in Figure 3, and their admissibility with terms of λ_{IR} can be shown with a function $\Gamma \vdash A \to \Delta \vdash_{\mathrm{S}} \Gamma \to \Delta \vdash A$. A renaming $\Gamma \vdash_{\mathrm{R}} \Delta$ is a substitution $\Gamma \vdash_{\mathrm{S}} \Delta$ that consists of only variables, and can also be defined explicitly in an inductive manner.

The equational theory of λ_{IR} extends that of the simply-typed lambda calculus with Rule $\Box$-β and Rule $\Box$-η, and is given in Figure 4 by omitting the standard rules. The renaming function ren used in Rule $\Box$-β renames a term using a function factor : $\Delta \lhd_{\lambda_{\mathrm{IR}}} \Gamma \to \Gamma \vdash_{\mathrm{R}} \Delta, \text{🔒}$ that constructs a renaming from a value of the modal accessibility relation.

$$\frac{}{\Gamma \vdash_{\mathrm{S}} \mathsf{empty} : \cdot} \qquad \frac{\Gamma \vdash_{\mathrm{S}} s : \Delta \qquad \Gamma \vdash t : A}{\Gamma \vdash_{\mathrm{S}} \mathsf{ext}\, s\, t : \Delta, A}$$

$$\frac{\Theta \vdash_{\mathrm{S}} s : \Delta \qquad e : \Theta \lhd_{\lambda_{\mathrm{IR}}} \Gamma}{\Gamma \vdash_{\mathrm{S}} \mathsf{ext}_{\text{🔒}}\, s\, e : \Delta, \text{🔒}}$$

Figure 3. Substitutions for λ_{IR}

$$\Box\text{-}\beta\ \frac{\Delta, \text{🔒} \vdash t : A \qquad e : \Delta \lhd_{\lambda_{\mathrm{IR}}} \Gamma}{\Gamma \vdash \mathsf{unbox}_{\lambda_{\mathrm{IR}}}\, (\mathsf{box}\, t)\, e \sim \mathsf{ren}\, (\mathsf{factor}\, e)\, t}$$

$$\Box\text{-}\eta\ \frac{\Gamma \vdash t : \Box A}{\Gamma \vdash t \sim \mathsf{box}\, (\mathsf{unbox}_{\lambda_{\mathrm{IR}}}\, t\, \mathsf{nil})}$$

Figure 4. Equational theory for λ_{IR} (omitting β- and η-equations for function types)

Evaluation in a Possible-World Model. We define the evaluation function for the modal fragment of λ_{IR} by induction on terms as follows. We omit the evaluation of the simply-typed fragment which is defined in the usual way.

$$\begin{aligned}
&[\![-]\!] : \Gamma \vdash A \longrightarrow (\forall w.\, [\![\Gamma]\!]_w \longrightarrow [\![A]\!]_w)\\
&[\![\ldots]\!]\, \gamma = \ldots\\
&[\![\mathsf{box}\, t]\!]\, \gamma = \lambda i.\, \lambda m.\, [\![t]\!]\, (m, \mathsf{wk}\, i\, \gamma)\\
&[\![\mathsf{unbox}_{\lambda_{\mathrm{IR}}}\, t\, e]\!]\, \gamma = [\![t]\!]\, \delta\, \mathsf{id}_{\mathrm{R}}\, m\\
&\qquad \text{where } (m, \delta) = \mathsf{trim}_{\lambda_{\mathrm{IR}}}\, \gamma\, e
\end{aligned}$$

To evaluate the term $\Gamma \vdash \mathsf{box}\, t : \Box A$ with $\gamma : [\![\Gamma]\!]_w$, we construct a function of the expected type $\forall w'.\, w\ R_i\ w' \longrightarrow \forall v.\, w'\ R_m\ v \longrightarrow [\![A]\!]_v$. Using the arguments $i : w\ R_i\ w'$ and $m : w'\ R_m\ v$, we recursively evaluate the term $\Gamma, 🔒 \vdash t : A$ in the extended environment $(m, \mathsf{wk}\, i\, \gamma) : [\![\Gamma, 🔒]\!]_v$. To evaluate the term $\Gamma \vdash \mathsf{unbox}_{\lambda_{\mathrm{IR}}}\, t\, e : A$, where $e : \Delta \lhd_{\lambda_{\mathrm{IR}}} \Gamma$ for some Δ, we recursively evaluate $\Delta \vdash t : \Box A$ with a new environment δ that discards the part of γ that is not needed for t. The function $\mathsf{trim}_{\lambda_{\mathrm{IR}}} : [\![\Gamma]\!]_w \longrightarrow \Delta \lhd_{\lambda_{\mathrm{IR}}} \Gamma \longrightarrow [\![\Delta, 🔒]\!]_w$ projects γ to produce $m : v'\ R_m\ w$ and $\delta : [\![\Delta]\!]_{v'}$. We then apply the result of recursive evaluation to the identity renaming id_{R} and the value m to return the desired result.

4 Applicative Functors of λ_{IR}

Clouston's categorical semantics of the type former $\Box$ in λ_{IR} identifies the class of applicative functors that have a left adjoint.

Let $(C, \mathrm{Box}, \mathrm{Lock}, \rho)$ be a model in the categorical semantics of λ_{IR}, i.e. a Cartesian-closed category C with an adjunction $\mathrm{Lock} \dashv \mathrm{Box}$ and a natural transformation $\rho : \mathrm{Id} \stackrel{.}{\longrightarrow} \mathrm{Box}$. The context operator 🔒 and the type former $\Box$ are interpreted by the adjoint functors Lock and Box, respectively, and the invertible rule $\Box$-Intro is interpreted by the (natural) bijection that sends a morphism $t : C(\mathrm{Lock}\, [\![\Gamma]\!], [\![A]\!])$ to its adjunct $\mathsf{box}\, t : C([\![\Gamma]\!], \mathrm{Box}\, [\![A]\!])$, which can be pictured similar to the rule like this:

$$\frac{C(\mathrm{Lock}\, [\![\Gamma]\!], [\![A]\!])}{C([\![\Gamma]\!], \mathrm{Box}\, [\![A]\!])}$$

This bijection implies additional properties for the applicative functors that are identified by the categorical semantics of λ_{IR}. Since the applicative functor Box has a left adjoint, it necessarily preserves binary products for instance, which not all applicative functors do. In fact, if we were to extend λ_{IR} with product types $A \times B$ in the usual way then we could define an inverse to the function $\Box(A \times B) \longrightarrow \Box A \times \Box B$.

In this section, we give three examples of applicative functors which do have a left adjoint, and can thus be used to interpret λ_{IR}.

Reader Monad. Let C be any Cartesian-closed category C and $E : C$ be any object. The exponentiation functor $X \mapsto E \Rightarrow X$ has the product functor $X \mapsto X \times E$ as its left adjoint and the constant morphisms $X \longrightarrow E \Rightarrow X$ constitute a natural transformation $\mathrm{Id} \stackrel{.}{\longrightarrow} E \Rightarrow -$.

$$\frac{C(X \times E, Y)}{C(X, E \Rightarrow Y)}$$

Redaction Monad. Let CSet be the category of classified sets [1, 12] over a given set $\mathcal{L}$ of "security levels". A classified set $\mathcal{X}$ is given by an underlying set X equipped with a family of reflexive binary relations R_ℓ on X indexed by "security levels" $\ell \in \mathcal{L}$. A function $f : X \longrightarrow Y$ between the underlying sets of two classified sets $\mathcal{X}$ and $\mathcal{Y}$ is a map $\mathcal{X} \longrightarrow \mathcal{Y}$ if and only if it is relation-preserving, i.e. if $f\,x\ R_\ell\ f\,x'$ whenever $x\ R_\ell\ x'$ for $x, x' \in X$ and $\ell \in \mathcal{L}$. Fix a subset $\pi \subseteq \mathcal{L}$ of "secret" security levels. The functor $\mathcal{X} \mapsto \mathrm{T}\,\mathcal{X}$ that replaces the relations R_ℓ at secret security levels $\ell \in \pi$ with the total relation on X has as its left adjoint the functor $\mathcal{X} \mapsto \mathrm{D}\,\mathcal{X}$ that replaces the relations R_ℓ for $\ell \in \pi$ instead with the identity relation. The identity functions $X \longrightarrow X$ are maps between the classified sets $\mathcal{X}$ and $\mathrm{T}(\mathcal{X})$ and hence constitute a natural transformation $\mathrm{Id} \stackrel{.}{\longrightarrow} \mathrm{T}$.

$$\frac{\mathrm{CSet}(\mathrm{D}\,\mathcal{X}, \mathcal{Y})}{\mathrm{CSet}(\mathcal{X}, \mathrm{T}\,\mathcal{Y})}$$

Later Modality. Let $\mathrm{PSh}(\omega)$ be the category of *contravariant* set-valued functors P, Q from the preorder

$$\omega : 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$$

of natural numbers $n \in \mathbb{N}$ and natural transformations $\alpha : P \stackrel{.}{\longrightarrow} Q$ between them. The functor $P \mapsto \blacktriangleright P$ that replaces the set $P(0)$ by a singleton set $\{*\}$ and the sets $P(n+1)$ by the sets $P(n)$ has as its left adjoint the functor $P \mapsto P\lhd$ that replaces the sets $P(n)$ by the sets $P(n+1)$. The restriction maps

$$P \stackrel{.}{\longrightarrow} \blacktriangleright P, p \in P(n) \mapsto \begin{cases} * & n = 0 \\ p|m & n = m+1 \end{cases}$$

constitute a natural transformation $\mathrm{Id} \stackrel{.}{\longrightarrow} \blacktriangleright -$.

$$\frac{\mathrm{PSh}(\omega)(P\lhd, Q)}{\mathrm{PSh}(\omega)(P, \blacktriangleright Q))}$$

When we see the (reverse) preorder ω^{op} as the type of worlds n, $m \in \mathbb{N}$ with the intuitionistic accessibility relation $n\ R_i\ m \Leftrightarrow n \geq m$ and equip it with the modal accessibility relation $n\ R_m\ m :\Leftrightarrow n\ \mathrm{isSucc}\ m \Leftrightarrow n = m + 1$ then we recover the later modality $\blacktriangleright$ and its left adjoint $\lhd$ as the necessity modality Box and its left adjoint Lock for the frame $(\mathbb{N}, \geq, \mathrm{isSucc})$, and the natural transformation $\mathrm{Id} \stackrel{.}{\longrightarrow} \blacktriangleright -$ as the implication $\mathrm{Id} \stackrel{.}{\longrightarrow} \mathrm{Box}$. Note that we indeed have $R_m\, ;\, R_i \subseteq R_i\, ;\, R_m$ and $R_m \subseteq R_i$ for these definitions.

5 Axiom R in Modal Logic

Unlike the intuitionistic modal logic IR, the axiomatization of a classical modal logic CR as an extension of the basic classical modal logic CK with axiom R exhibits the validity of several unexpected formulae. For example, contrary to what the axiomatization of CR might suggest, we can also show that $\Box\Box A \Rightarrow \Box A$ for an arbitrary formula A.

In this section, we analyze the interpretation of axiom R in possible-world models of classical modal logic. We observe that the frame condition on R_m required for validating axiom R in classical logic is too restrictive since the models that satisfy this condition are degenerate. This is unlike the case of axioms T and 4, whose classical and intuitionistic frame conditions on R_m coincide. We discuss this seeming disruption in harmony for axiom R and show that the classical frame condition—although not identical—is in fact a special case of the intuitionistic one.

We work in a classical metatheory in this section. Moreover, as is usual in classical modal logic, we work with proof-irrelevant propositions and relations, and hence we drop all the coherence conditions mentioned previously.

Classical Possible-World Semantics. A possible-world model in classical modal logic is defined as before by omitting the R_i relation and the frame conditions that mention it. It is thus given by a frame F defined as a tuple (W, R_m) and a valuation V_ι for the atomic formula ι. The interpretation of a formula $\Box A$ at a world w is given as:

$$[\![\Box A]\!]_w = \forall v.\, w\ R_m\ v \longrightarrow [\![A]\!]_v$$

The basic classical modal logic CK, the classical variant of IK, is both sound and complete with respect to possible-world models under this interpretation. That is, for a sequent $\Gamma \vdash A$ provable in CK, we have that $\Gamma \vDash A$, i.e. $[\![\Gamma]\!]_w \longrightarrow [\![A]\!]_w$ for every world w in all classical possible-world models (soundness), and vice versa (completeness).

Frame Correspondence. A modal axiom is said to *correspond* to a frame condition (on R_m) if the condition is both necessary and sufficient for the validation of the axiom in the possible-world semantics. The modal axiom T corresponds to reflexivity of R_m, while 4 corresponds to transitivity. The respective modal logics CT and CK4 are also complete with respect to the models that satisfy the frame conditions that the axioms correspond to.

While axiom R has a corresponding frame condition that identifies models where worlds are isolated with respect to R_m ($w\ R_m\ v$ implies $w = v$ for all $w, v : W$), this condition is too restrictive. Models that satisfy this condition also validate the formulas $\Box\Box A \Rightarrow \Box A$ for instance. This is because, given $[\![\Box\Box A]\!]_w$ at some world w, any neighbouring world of w must be w itself, and thus $[\![\Box A]\!]_w$ also holds.

Classical vs. Intuitionistic Modal Logic. The possible-world models for a classical modal logic can be recovered from the intuitionistic definition given earlier in Section 2 by fixing R_i to be the identity relation ($w\ R_i\ v$ if and only if $w = v$). This means that the classical possible-world models are a subclass of intuitionistic possible-world models, and that the classical models must at least satisfy the conditions satisfied by the intuitionistic models. In particular, this means that a corresponding frame condition on R_m used to interpret a classical modal logic must at least be the one used for its intuitionistic counterpart.

For the logics IT, IK4 and IS4, the corresponding frame conditions on R_m are in fact identical to their classical counterparts. For the logic IR, on the other hand, observe that the corresponding frame condition $R_m \subseteq R_i$, in combination with defining R_i as the identity relation, does in fact yield the condition that $w\ R_m\ v$ implies $w = v$.

6 Further and Related Work

Fitch-Style Calculi. The basic Fitch-style calculus λ_{IK} was originally presented by Borghuis [3] and Martini and Masini [13], and its categorical semantics were later identified by Clouston [5]. The calculus λ_{IS4}, on the other hand, has several different formulations [6, 7, 5, 9], where the primary difference lies in whether the logical equivalence $\Box A \Leftrightarrow \Box\Box A$ can also be shown to be an isomorphism, i.e. whether in the semantics the comonad $\Box$ is also *idempotent*. The calculi λ_{IT} and λ_{IK4} were presented explicitly by Valliappan, Ruch, and Tomé Cortiñas [18], but have also been alluded to in previous work [6, 7, 5]. Clouston presented the calculus λ_{IR} and its categorical semantics, albeit with a stronger variant of Rule $\Box$-β: unbox (box t) $\sim t$.

Valliappan, Ruch, and Tomé Cortiñas presented the possible-world semantics for the calculi λ_{IK}, λ_{IK4}, λ_{IT} and λ_{IS4}, and also proved normalization for these calculi by constructing NbE models as instances. In contrast to earlier work, this approach bypasses syntactic considerations in the implementation of the normalization algorithm. In this article, we investigate the possible-world semantics of λ_{IR} and suggest the construction of an NbE model for it. Normalization for λ_{IR} was also considered by Clouston, but without Rule $\Box$-η.

Towards a Fitch-Style Calculus for Monads. Two of the three examples in Section 4 are in fact monads, which suggests that it might be worthwhile formulating a Fitch-style monad calculus which incorporates the following axiom:

$$\mathrm{J} : \Box\Box A \Rightarrow \Box A$$

Such a calculus would allow us to understand the possible-world interpretation of a monad as a necessity modality.

We define a Fitch-style calculus $\lambda_{\mathrm{PLLBox}}$ (for "Propositional Lax Logic" [8] formulated using a necessity modality "Box") similar to λ_{IR} in Section 3 by replacing the modal accessibility relation $\lhd_{\lambda_{\mathrm{IR}}}$ in the $\Box$-elimination rule with the relation $\lhd_{\lambda_{\mathrm{PLLBox}}}$ defined in Figure 5.

$$\frac{}{\text{base-var} : \Gamma, A \lhd_{\lambda_{\text{PLLBox}}} \Gamma, A, \text{🔒}}$$

$$\frac{}{\text{base-lock} : \Gamma, \text{🔒} \lhd_{\lambda_{\text{PLLBox}}} \Gamma, \text{🔒}} \qquad \frac{e : \Delta \lhd_{\lambda_{\text{PLLBox}}} \Gamma}{\text{var}\, e : \Delta \lhd_{\lambda_{\text{PLLBox}}} \Gamma, A}$$

$$\frac{e : \Delta \lhd_{\lambda_{\text{PLLBox}}} \Gamma}{\text{lock}\, e : \Delta \lhd_{\lambda_{\text{PLLBox}}} \Gamma, \text{🔒}}$$

Figure 5. Modal accessibility relation on contexts (λ_{PLLBox})

In classical modal logic, the axiom J corresponds to the following frame condition:

$$R_m \subseteq R_m \,;\, R_m$$

The relation $\lhd_{\lambda_{\text{PLLBox}}}$ satisfies this condition, indicating that it is likely the right choice for λ_{PLLBox}. The equational theory of λ_{PLLBox} and its categorical semantics requires further study. This calculus is expected to inevitably exhibit *strength* (a natural transformation $A \times \Box B \overset{\cdot}{\rightarrow} \Box(A \times B)$) similar to the case for λ_{IR}. The possible-world interpretation and NbE model construction is also likely to resemble the process for λ_{IR}, but we have not yet investigated this.

Beyond Fitch-Style Calculi. Several examples of applicative functors that are also monads do not possess a left adjoint, thus indicating the limited applicability of the Fitch-style calculus for modelling monads. To study the possible-world semantics of the applicative functors and monads exempted by the Fitch-style calculi, it might be better to use other formulations, such as Moggi's monadic metalanguage [16] for monads, or the more general *dual context-calculi* [17, 11] and *multi-modal type theory* [10]. The possible-world interpretation of many of these calculi remain an open problem, thus leaving much in the avenue of future work.

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