Cover semantics beyond distributivity

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Abstract

Cover semantics is an approach to modelling non-classical logics, pioneered by Goldblatt, that can be understood as a generalisation of Beth-Kripke-Joyal semantics for intuitionistic propositional logic. The versatility of cover semantics has been demonstrated in recent results that show how cover semantics can be used to constructively prove meta-logical results such as completeness, consistency and normalisation for a variety of logics with an intuitionistic base. In this work, we take a step backwards and show that this versatility runs deeper than previously shown. We show that cover semantics can as well be used for logics that forego the intuitionistically characteristic distribution of conjunction over disjunction. We begin with cover semantics for a non-distributive positive logic, i.e. without implication or negation, known as lattice logic and show how it can be extended to model complex systems with a lattice base including a positive modal logic, first-order lattice logic, and a typed program calculus with commuting conversions that corresponds to lattice logic.

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1 Introduction

Cover systems provide an alternative approach for modelling the truth of logical formulas, compared to the widely used truth-functional approach of classical propositional logic and the relational approach of Kripke's semantics for intuitionistic and modal logics [37, 38]. The origins of cover semantics can be traced to the topological notion of *local truth*, where a logical formula is locally true at a world w when w has a *cover*, e.g. an open neighbourhood, in which the formula is true. This leads to a "non-local" interpretation for disjunctions:

$$w \models \varphi \vee \psi \quad \text{iff} \quad \text{there exists a cover of } w \text{ in which every element satisfies } \varphi \text{ or } \psi.$$

The key idea underlying cover semantics is to let go of the rigid structural assumption that covers must be open neighbourhoods in some topological space, and instead define them, for a given set W of worlds, simply as a collection of subsets of W for each world $w \in W$.

The use of cover systems dates back to Joyal [36], and the so-called *Kripke-Joyal semantics* for intuitionistic logic in topoi [40]. It was later popularised by Goldblatt, who initially used it to model a first-order logic characterised by quantales [23] using an interpretation for disjunctions reminiscent of Beth's semantics [5]. Cover semantics has since been shown to be a remarkably versatile approach for modelling a variety of non-classical logics including intuitionistic modal logics [24, 46, 50], relevance logic [25], and substructural logics [28, 26].

A central theme in Goldblatt's work on cover semantics is to give representations for the algebraic models of non-classical logics. The starting point is a cover system $(W, \leq, \lhd)$, where W is a set of worlds, $\leq$ a preorder on W, and $\lhd \subseteq W \times \mathcal{P}(W)$ a cover relation on W.

A subset $A \subseteq W$ is called an *upset* when $w \in A$ and $w \leq v$ imply $v \in A$, and *localised* when $w \lhd \alpha \subseteq A$ implies $w \in A$. Under suitable conditions, Goldblatt shows that the collection of localised upsets of a cover system represent algebras such as locales and Heyting algebras.

Despite its mathematical elegance, Goldblatt's top-down representation-led approach to cover semantics is not always amenable to standard techniques used to constructively analyse a logic and its proof system. For example, Goldblatt's cover semantics for quantified lax logic does not support the construction of a Henkin-style model without prime filters [24, Section 8] even though it is shown sound and complete via a representation theorem. This is unfortunate, because cover semantics is exceptionally well suited for constructive proofs. Recent work by Valliappan [50] provides cover semantics amenable to constructive completeness proofs for propositional intuitionistic logic as well as several modal extensions, but their work is limited to distributive logics. Remedying this limitation is the starting point of our work.

In this paper we revisit cover semantics for (not necessarily distributive) lattices, which models the logic known as lattice logic [45], non-distributive positive logic [11, 13], finite sum-product logic [12] and weak positive logic [7]. This logic originates from Dalla Chiara [11], in the context of quantum logics. Subsequently, non-distributive logics have been studied extensively, from the perspective of proof theory [11, 45, 12], semantics [49, 45, 19, 13, 29], and duality theory [49, 35, 34, 7, 6]. Besides quantum logic, non-distributivity is also a common phenomenon in theoretical computer science [21, 43] and linguistics [39, 51].

Our work on lattice logic improves upon Goldblatt's representation of lattices [27] in two prominent ways. First, we minimise the definition of a cover system for lattice logic by stripping the preorder. This liberates the construction of models from routine overhead, and allows us to unveil an embedding of lattice logic into a suitable classical monotone modal logic, which is simpler than existing translations that use two-sorted semantics [33, 14]. Second, we leverage the convenience of our reformulation for constructively proving completeness and normalisation. After setting up this propositional basis, we extend our development in three different directions to illustrate the power of cover semantics to surpass existing results. We prove completeness constructively for three important extensions, namely with modal operators, quantifiers and predicates, and proof terms accompanied by an equational theory.

1. Modal lattice logic. In Section 3, we show that cover semantics for lattice logic can be extended to accommodate modal operators. We begin by adding a monotone modal operator $\heartsuit$ to lattice logic and interpret the resulting logic using a second cover relation $\blacktriangleleft$. A unique outcome of this study is a constructive proof of completeness for a positive modal logic. We then show how additional modal axioms can be handled in the semantics.

2. First-order lattice logic. Next, in Section 4, we enrich lattice logic with constants, predicates and quantifiers and show how cover systems adapt to this extension. By adapting Goldblatt's original proof of completeness for first-order intuitionistic logic [24] to the setting of lattice logic, we show that cover systems can also constructively model first-order logics without the distributivity of conjunction over disjunction.

3. Program calculus. Finally, in Section 5, we define and analyse a typed program calculus $\mathbb{LC}$ based on lattice logic, and present a proof-relevant refinement of cover semantics that models its equational theory. We identify a categorical semantics for $\mathbb{LC}$ that validates *commuting conversions* and show that the refined cover semantics of $\mathbb{LC}$ is an instance of it. Cover models notably allow us to conduct an independent and isolated analysis of commuting conversions for sum types in the absence of functions (which force distributivity), untangling a notoriously difficult combination in the study of typed lambda calculi [1, 18, 41].

2 Lattice logic

Lattice logic is a "positive" sublogic of intuitionistic logic, i.e. a logic that omits implication and negation. Formally, we define the language $\mathbf{L}$ and contexts $\mathsf{Ctx}_{\mathbf{L}}$ by:

$$\varphi ::= p \in \text{Prop} \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \qquad\qquad \Gamma ::= \cdot \mid \Gamma, \varphi$$

As usual, $\top$ denotes truth, $\bot$ denotes falsity, $\wedge$ denotes conjunction and $\vee$ denotes disjunction. The connectives $\wedge$ and $\vee$ have equal operator precedence and associate to the right. Note that a context Γ is simply a finite list of formulas, with $\cdot$ denoting the empty list.

▶ **Definition 1.** A sequent-style natural deduction proof system for lattice logic $\mathcal{L}$ can be given using the following inference rules. A *judgment* $\Gamma \vdash \varphi$ asserts that formula φ has a proof under the assumption that all formulas in context Γ have a proof. We sometimes write $\Gamma \vdash_{\mathcal{L}} \varphi$ to emphasise that it is a judgment in the proof system of lattice logic.

$$\frac{\varphi \in \Gamma}{\Gamma \vdash \varphi}\ (\textsc{hyp}) \qquad \frac{}{\Gamma \vdash \top}\ (\top\text{-}\textsc{in}) \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ (\bot\text{-}\textsc{el})$$

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\ (\wedge\text{-}\textsc{el}_1) \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}\ (\wedge\text{-}\textsc{el}_2) \qquad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}\ (\wedge\text{-}\textsc{in})$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ (\vee\text{-}\textsc{in}_1) \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ (\vee\text{-}\textsc{in}_2) \qquad \frac{\Gamma \vdash \varphi \vee \psi \quad \varphi \vdash \chi \quad \psi \vdash \chi}{\Gamma \vdash \chi}\ (\vee\text{-}\textsc{el})$$

For $\Delta \in \mathsf{Ctx}_{\mathbf{L}}$, we write $\Gamma \vdash \Delta$ to denote that $\Gamma \vdash \varphi_i$ for all $\varphi_i \in \Delta$.

The inference rules consist of introduction (-IN) and elimination (-EL) rules. The absence of context in some of the premises of the ($\vee$-EL) rule prevents the rule from proving distributivity.

▶ **Lemma 2** (Cut admissibility). *If $\Gamma \vdash \Delta$ and $\Delta \vdash \varphi$ then $\Gamma \vdash \varphi$.*

Proof. By induction on the derivation of the judgment $\Delta \vdash \varphi$. ◀

2.1 Cover models for lattice logic

The cover systems used in this paper arise from those in [27, Section 2] by leaving out the preorder and the refinement condition.

▶ **Definition 3.** A *cover system* for lattice logic is a tuple $(W, \lhd)$ consisting of a set W of "worlds" and a *cover relation* $\lhd \subseteq W \times \mathcal{P}(W)$ that satisfies the following conditions:

$$w \lhd \{w\} \tag{identity}$$

$$\text{if } w \lhd \alpha \text{ and } v \lhd \alpha_v \text{ for all } v \in \alpha, \text{ then } w \lhd \bigcup_{v \in \alpha} \alpha_v \tag{transitivity}$$

For a set $\alpha \subseteq W$ and world $w \in W$, we say α *covers* w, or w is *covered by* α, when $w \lhd \alpha$. We say that a set A is *localised* if it contains all worlds that are covered by some subset of A, and write $\text{Loc}(W, \lhd)$ for the collection of localised subsets of $(W, \lhd)$.

A cover α of a world w may be understood as a "locality" of knowledge states capturing knowledge local to w. A localised subset can then be thought of as a set closed under local knowledge. Closing a subset under locality yields the *locality operator* $\langle\lhd\rangle$ on $\mathcal{P}(W)$, given by $\langle\lhd\rangle A := \{w \in W \mid w \lhd \alpha \subseteq A \text{ for some } \alpha\}$. A set $A \subseteq W$ is localised if $\langle\lhd\rangle A \subseteq A$. Moreover, $\langle\lhd\rangle$ is *monotonic* (i.e. $A \subseteq B$ implies $\langle\lhd\rangle A \subseteq \langle\lhd\rangle B$), and (identity) and (transitivity) entail that it is *inflationary* (i.e. $A \subseteq \langle\lhd\rangle A$) and *idempotent* (i.e. $\langle\lhd\rangle\langle\lhd\rangle A = \langle\lhd\rangle A$). We have:

▶ **Lemma 4.** *Let $(W, \lhd)$ be a cover system.*

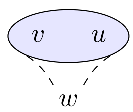
1. *The operator $\langle\lhd\rangle$ is monotonic, inflationary and idempotent, hence a closure operator.*
2. *For any $A \subseteq W$, the set $\langle\lhd\rangle A$ is localised.*
3. *The intersection of an arbitrary collection of localised subsets of $(W, \lhd)$ is localised.*

▶ **Definition 5.** A *cover model* for lattice logic is a triple $\mathfrak{M} = (W, \lhd, V)$ consisting of a cover system $(W, \lhd)$ and a valuation $V : \mathrm{Prop} \to \mathrm{Loc}(W, \lhd)$ of propositional atoms as localised upsets of $(W, \lhd)$. The truth of formulas at a world w is defined recursively as follows:

$$\begin{array}{lll}
\mathfrak{M}, w \models p & \text{iff} & w \in V(p) \\
\mathfrak{M}, w \models \top & & \text{always} \\
\mathfrak{M}, w \models \bot & \text{iff} & w \lhd \emptyset \\
\mathfrak{M}, w \models \varphi \wedge \psi & \text{iff} & \mathfrak{M}, w \models \varphi \text{ and } \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models \varphi \vee \psi & \text{iff} & \text{there exists } \alpha \subseteq W \text{ such that } w \lhd \alpha \subseteq [\![\varphi]\!]^{\mathfrak{M}} \cup [\![\psi]\!]^{\mathfrak{M}}
\end{array}$$

Here $[\![\varphi]\!]^{\mathfrak{M}} := \{w \in W \mid \mathfrak{M}, w \models \varphi\}$ denotes the *truth set* of φ. We extend this to contexts via $[\![\Gamma]\!]^{\mathfrak{M}} = \bigcap_{\psi\in\Gamma} [\![\psi]\!]^{\mathfrak{M}}$, and drop the superscript $\mathfrak{M}$ when it is clear from context. We write $\Gamma \models^{\mathfrak{M}} \varphi$ when $[\![\Gamma]\!]^{\mathfrak{M}} \subseteq [\![\varphi]\!]^{\mathfrak{M}}$, and $\Gamma \models_{\mathcal{L}} \varphi$ when $\Gamma \models^{\mathfrak{M}} \varphi$ for all cover models $\mathfrak{M}$.

▶ **Example 6.** Consider the cover model $(W, \lhd, V)$ with $W = \{w, v, u\}$ and with $\lhd$ given by $w \lhd \{v, u\}$ and $x \lhd \{x\}$ for all $x \in W$. Define $V(p) = \{w\}$, $V(q) = \{v\}$ and $V(r) = \{u\}$. Then $w \models p \wedge (q \vee r)$ but $w \not\models (p \wedge q) \vee (p \wedge r)$, so the model invalidates distributivity.



A straightforward induction on the structure of φ proves the following analogue of the persistence or heredity property from intuitionistic logic.

▶ **Lemma 7.** *For any cover model $\mathfrak{M}$ and formula $\varphi \in \mathbf{L}$ the set $[\![\varphi]\!]^{\mathfrak{M}}$ is localised.*

▶ **Proposition 8** (Soundness). *For all $\Gamma \in \mathsf{Ctx}_{\mathbf{L}}$ and $\varphi \in \mathbf{L}$, $\Gamma \vdash \varphi$ implies $\Gamma \models_{\mathcal{L}} \varphi$.*

Proof. By induction on the derivation of the judgment $\Gamma \vdash_{\mathcal{L}} A$, we show $[\![\Gamma]\!]^{\mathfrak{M}} \subseteq [\![\varphi]\!]^{\mathfrak{M}}$ for an arbitrary model $\mathfrak{M}$. The interesting cases are that of ($\bot$-EL) and ($\vee$-EL). For ($\bot$-EL), the induction hypothesis gives us $[\![\Gamma]\!] \subseteq [\![\bot]\!]$. Since $[\![\bot]\!] = \langle\lhd\rangle\emptyset \subseteq \langle\lhd\rangle[\![\varphi]\!]$ by Lemma 4(1), we can conclude $\Gamma \subseteq [\![\varphi]\!]$ as desired, by invoking Lemma 7 for formula φ which gives us $\langle\lhd\rangle[\![\varphi]\!] \subseteq [\![\varphi]\!]$. A similar argument applies for ($\vee$-EL) by applying Lemma 7 on the concluding formula χ. ◀

2.2 Completeness

We prove completeness of $\mathcal{L}$ with respect to cover semantics by constructing a canonical model $\mathfrak{M}_{\mathcal{L}}$ in which entailment ($\Gamma \models^{\mathfrak{M}_{\mathcal{L}}} \varphi$) coincides with derivability ($\Gamma \vdash_{\mathcal{L}} \varphi$).

▶ **Definition 9.** For $\varphi \in \mathbf{L}$, we denote by $\mathsf{Ant}(\varphi) := \{\Gamma \in \mathsf{Ctx} \mid \Gamma \vdash_{\mathcal{L}} \varphi\}$ the set of *antecedents* of a formula φ. Define the relation $\lhd_{\mathcal{L}} \subseteq \mathsf{Ctx} \times \mathcal{P}(\mathsf{Ctx})$ inductively by:

$$\frac{}{\Gamma \lhd_{\mathcal{L}} \{\Gamma\}}\ (\lhd\text{-ID}) \qquad \frac{\Gamma \in \mathsf{Ant}(\bot)}{\Gamma \lhd_{\mathcal{L}} \emptyset}\ (\lhd\bot) \qquad \frac{\Gamma \in \mathsf{Ant}(\varphi \vee \psi) \quad \{\varphi\} \lhd_{\mathcal{L}} \alpha_1 \quad \{\psi\} \lhd_{\mathcal{L}} \alpha_2}{\Gamma \lhd_{\mathcal{L}} \alpha_1 \cup \alpha_2}\ (\lhd\vee)$$

Observe that this relation is "proof-relevant" in the sense that a relationship $\Gamma \lhd_{\mathcal{L}} \alpha$ is witnessed by a derivation tree constructed using the above inference rules.

▶ **Lemma 10.** $(\mathsf{Ctx}_{\mathbf{L}}, \lhd_{\mathcal{L}})$ *is a cover system, and* $\mathsf{Ant}(\varphi) \in \mathrm{Loc}(\mathsf{Ctx}_{\mathbf{L}}, \lhd_{\mathcal{L}})$ *for all* $\varphi \in \mathbf{L}$.

Define the canonical valuation $V_{\mathcal{L}}$ for $(\mathsf{Ctx}_{\mathbf{L}}, \lhd_{\mathcal{L}})$ by $V_{\mathcal{L}}(p) = \mathsf{Ant}(p)$. Then $\mathfrak{M}_{\mathcal{L}} = (\mathsf{Ctx}_{\mathbf{L}}, \lhd_{\mathcal{L}}, V_{\mathcal{L}})$ is a cover model. We can now prove that the set of antecedents of any $\varphi \in \mathbf{L}$ is identical to its truth set in $\mathfrak{M}_{\mathcal{L}}$, i.e. $\mathsf{Ant}(\varphi) = [\![\varphi]\!]^{\mathfrak{M}_{\mathcal{L}}}$. In other words:

▶ **Lemma 11** (Truth lemma)**.** *For any $\varphi \in \mathbf{L}$ and $\Gamma \in \mathsf{Ctx}_{\mathbf{L}}$, we have $\Gamma \vdash \varphi$ iff $\mathfrak{M}_{\mathcal{L}}, \Gamma \models \varphi$.*

Proof. We use induction on the structure of φ. The cases for $\varphi = \top$ and $\varphi = p \in \mathrm{Prop}$ hold by definition, and that for $\varphi = \psi \wedge \chi$ is routine. If $\varphi = \bot$ and $\Gamma \vdash_{\mathcal{L}} \bot$ then $\Gamma \lhd_{\mathcal{L}} \emptyset \subseteq [\![\bot]\!]$, so $\mathfrak{M}_{\mathcal{L}}, \Gamma \models \bot$. Conversely, if $\mathfrak{M}_{\mathcal{L}}, \Gamma \models \bot$ then $\Gamma \in \langle \lhd_{\mathcal{L}} \rangle \emptyset$, hence by monotonicity of $\langle \lhd_{\mathcal{L}} \rangle$ and localisation of $\mathsf{Ant}(\bot)$, we have $\Gamma \in \langle \lhd_{\mathcal{L}} \rangle (\mathsf{Ant}(\bot)) \subseteq \mathsf{Ant}(\bot)$. This implies $\Gamma \vdash \bot$.

Lastly, if $\varphi = \psi \vee \chi$ and $\Gamma \vdash_{\mathcal{L}} \psi \vee \chi$ then ($\lhd$-ID) and induction entail $\{\psi\} \lhd_{\mathcal{L}} \{\{\psi\}\} \subseteq [\![\psi]\!]$, and $\{\chi\} \lhd_{\mathcal{L}} \{\{\chi\}\} \subseteq [\![\chi]\!]$. Therefore ($\lhd\vee$) gives $\Gamma \lhd_{\mathcal{L}} \{\{\psi\}, \{\chi\}\} \subseteq [\![\psi]\!] \cup [\![\chi]\!]$, so by definition $\mathfrak{M}_{\mathcal{L}}, \Gamma \models \psi \vee \chi$. For the converse, assume $\mathfrak{M}_{\mathcal{L}}, \Gamma \models \psi \vee \chi$. Then by definition, induction, monotonicity of $\langle \lhd_{\mathcal{L}} \rangle$ and the fact that $\mathsf{Ant}(\psi \vee \chi)$ is localised, we find $\Gamma \in \langle \lhd_{\mathcal{L}} \rangle ([\![\psi]\!] \cup [\![\chi]\!]) = \langle \lhd_{\mathcal{L}} \rangle (\mathsf{Ant}(\psi) \cup \mathsf{Ant}(\chi)) \subseteq \langle \lhd_{\mathcal{L}} \rangle (\mathsf{Ant}(\psi \vee \chi)) \subseteq \mathsf{Ant}(\psi \vee \chi)$. So $\Gamma \vdash \psi \vee \chi$. ◀

▶ **Theorem 12.** *For any $\Gamma \in \mathsf{Ctx}_{\mathbf{L}}$ and $\varphi \in \mathbf{L}$, we have $\Gamma \models_{\mathcal{L}} \varphi$ if and only if $\Gamma \vdash_{\mathcal{L}} \varphi$.*

Proof. The direction from right to left is soundness. For the converse, note that for each $\psi \in \Gamma$, combining (HYP) and Lemma 11 gives $\mathfrak{M}_{\mathcal{L}}, \Gamma \models \psi$. Therefore $\Gamma \in \bigcap_{\psi \in \Gamma} [\![\psi]\!]$, so that the assumption $\Gamma \models_{\mathcal{L}} \varphi$ entails $\Gamma \in [\![\varphi]\!]$, i.e. $\mathfrak{M}_{\mathcal{L}}, \Gamma \models \varphi$, hence $\Gamma \vdash_{\mathcal{L}} \varphi$ by Lemma 11. ◀

▶ Remark 13. Cover systems can be viewed as neighbourhood frames. As such, they provide semantics for classical monotone modal logic [10, 31, 44], with a modality $\bigcirc$ interpreted by $w \models \bigcirc\varphi$ iff $w \lhd \alpha \subseteq [\![\varphi]\!]$ for some cover α. It can be shown that the extension of classical monotone modal logic with the S4-axioms $p \to \bigcirc p$ and $\bigcirc\bigcirc p \to \bigcirc p$ is sound and complete with respect to the class of cover systems. Let us call this logic $\mathcal{MS}4$. We can translate $\mathbf{L}$ into the classical modal language via

$$t(p) := \bigcirc p, \quad t(\top) := \top, \quad t(\bot) := \bigcirc\bot, \quad t(\varphi \wedge \psi) := t(\varphi) \wedge t(\psi), \quad t(\varphi \vee \psi) := \bigcirc(t(\varphi) \vee t(\psi))$$

and extend this to contexts as expected. It can then be shown that for all $\varphi \in \mathbf{L}$ and $\Gamma \in \mathsf{Ctx}_{\mathbf{L}}$ we have $\Gamma \vdash \varphi$ if and only if $t(\Gamma) \vdash_{\mathcal{MS}4} t(\varphi)$ (see Appendix A). In other words, lattice logic can be embedded into the classical monotone modal logic $\mathcal{MS}4$.

2.3 Normalisation by Evaluation

In this subsection, we construct a different kind of canonical model to show that every judgement derivable in lattice logic has a derivation in normal form. Our model construction closely follows the semantics-based normalisation technique known as Normalisation by Evaluation (NbE) [4, 3]. The normal forms defined below serve the same purpose as a "cut-free" proof in sequent calculus and ensure that there are no unnecessary inference rules.

▶ **Definition 14.** A judgement $\Gamma \vdash_{\mathrm{NF}} \varphi$ characterises derivations in normal form, while $\Gamma \vdash_{\mathrm{NE}} \varphi$ characterises derivations in neutral form. The judgement $\Gamma \vdash_{\mathrm{NE}} \varphi$ ensures that the formula φ is a subformula of some formula in Γ, while the judgement $\Gamma \vdash_{\mathrm{NF}} \varphi$ ensures that its derivation respects the subformula property, which means that at every inference step of its derivation the formulas in the premises are subformulas of some formula in the conclusion.

$$\frac{\varphi \in \Gamma}{\Gamma \vdash_{\mathrm{NE}} \varphi}\ (\textsc{hyp}) \quad \frac{\Gamma \vdash_{\mathrm{NE}} \varphi \wedge \psi}{\Gamma \vdash_{\mathrm{NE}} \varphi}\ (\wedge\text{-}\textsc{el}_1) \quad \frac{\Gamma \vdash_{\mathrm{NE}} \varphi \wedge \psi}{\Gamma \vdash_{\mathrm{NE}} \psi}\ (\wedge\text{-}\textsc{el}_2) \quad \frac{\Gamma \vdash_{\mathrm{NE}} p}{\Gamma \vdash_{\mathrm{NF}} p}\ (\textsc{Prop-emb})$$

$$\frac{}{\Gamma \vdash_{\mathrm{NF}} \top}\ (\top\text{-}\textsc{in}) \quad \frac{\Gamma \vdash_{\mathrm{NE}} \bot}{\Gamma \vdash_{\mathrm{NF}} \varphi}\ (\bot\text{-}\textsc{el}) \quad \frac{\Gamma \vdash_{\mathrm{NF}} \varphi \quad \Gamma \vdash_{\mathrm{NF}} \psi}{\Gamma \vdash_{\mathrm{NF}} \varphi \wedge \psi}\ (\wedge\text{-}\textsc{in}) \quad \frac{\Gamma \vdash_{\mathrm{NF}} \varphi}{\Gamma \vdash_{\mathrm{NF}} \varphi \vee \psi}\ (\vee\text{-}\textsc{in}_1)$$

$$\frac{\Gamma \vdash_{\mathrm{NF}} \psi}{\Gamma \vdash_{\mathrm{NF}} \varphi \vee \psi}\ (\vee\text{-}\textsc{in}_2) \qquad \frac{\Gamma \vdash_{\mathrm{NE}} \varphi \vee \psi \quad \varphi \vdash_{\mathrm{NF}} \chi \quad \psi \vdash_{\mathrm{NF}} \chi}{\Gamma \vdash_{\mathrm{NF}} \chi}\ (\vee\text{-}\textsc{el})$$

For a formula $\varphi \in \mathbf{L}$, let us write $\mathsf{Ne}(\varphi) := \{\Gamma \in \mathsf{Ctx} \mid \Gamma \vdash_{\mathrm{NE}} \varphi\}$ for the set of neutral antecedents of φ and $\mathsf{Nf}(\varphi) := \{\Gamma \in \mathsf{Ctx} \mid \Gamma \vdash_{\mathrm{NF}} \varphi\}$ for the set of normal antecedents of φ.

Recall that the essential character of the canonical model $\mathfrak{M}_{\mathcal{L}}$ used to prove completeness for lattice logic was its ability to equate entailment in the model $\mathfrak{M}_{\mathcal{L}}$ with the derivability of the corresponding judgment. In contrast, we now desire a canonical model $\mathfrak{M}_{\mathcal{N}}$ in which entailment in $\mathfrak{M}_{\mathcal{N}}$ implies derivability of the corresponding judgment in normal form, i.e. $\Gamma \models^{\mathfrak{M}_{\mathcal{N}}} \varphi$ implies $\Gamma \vdash_{\mathrm{NF}} \varphi$. We will achieve this by proving a refinement of the truth lemma characteristic of NbE algorithms known as *reification*, which states that $[\![\varphi]\!]^{\mathfrak{M}_{\mathcal{N}}} \subseteq \mathsf{Nf}(\varphi)$, and *reflection*, which states that $\mathsf{Ne}(\varphi) \subseteq [\![\varphi]\!]^{\mathfrak{M}_{\mathcal{N}}}$.

Define a canonical relation $\lhd_{\mathcal{N}} \subseteq \mathsf{Ctx} \times \mathcal{P}(\mathsf{Ctx})$ akin to the canonical relation $\lhd_{\mathcal{L}}$ from earlier by replacing all occurrences of Ant in the premises of the rules with Ne as follows:

$$\frac{}{\Gamma \lhd_{\mathcal{N}} \{\Gamma\}}\ (\lhd\text{-}\textsc{id}) \quad \frac{\Gamma \in \mathsf{Ne}(\bot)}{\Gamma \lhd_{\mathcal{N}} \emptyset}\ (\lhd\bot) \quad \frac{\Gamma \in \mathsf{Ne}(\varphi \vee \psi) \quad \{\varphi\} \lhd_{\mathcal{N}} \alpha_1 \quad \{\psi\} \lhd_{\mathcal{L}} \alpha_2}{\Gamma \lhd_{\mathcal{N}} \alpha_1 \cup \alpha_2}\ (\lhd\vee)$$

As before we can show that $\lhd_{\mathcal{N}}$ is a cover relation by repeating the induction on its construction. In light of this cover relation, we can further show that the set $\mathsf{Nf}(\varphi)$ is a localised set of the cover system $(\mathsf{Ctx}, \lhd_{\mathcal{N}})$ for any formula $\varphi \in \mathbf{L}$. This means we can define the canonical valuation by $V_{\mathcal{N}}(p) = \mathsf{Nf}(p)$, equating the truth of an atom with its normal antecedents. Let us now define a canonical NbE model $\mathfrak{M}_{\mathcal{N}} = (\mathsf{Ctx}_{\mathbf{L}}, \lhd_{\mathcal{N}}, V_{\mathcal{N}})$.

▶ **Lemma 15** (Reify-Reflect lemma). *For any $\varphi \in \mathbf{L}$, $[\![\varphi]\!]^{\mathfrak{M}_{\mathcal{N}}} \subseteq \mathsf{Nf}(\varphi)$ and $\mathsf{Ne}(\varphi) \subseteq [\![\varphi]\!]^{\mathfrak{M}_{\mathcal{N}}}$.*

Proof. By repeating the induction from Lemma 11 on the structure of φ. ◀

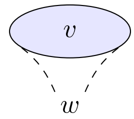
▶ **Theorem 16** (Completeness of normal forms for lattice logic). *$\Gamma \vdash_{\mathcal{L}} \varphi$ if and only if $\Gamma \vdash_{\mathrm{NF}} \varphi$.*

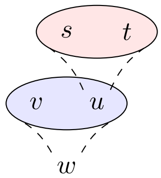
Proof. From left to right, note that for all $\psi \in \Gamma$ we have $\Gamma \in \mathsf{Ne}(\psi)$ by the $(\textsc{hyp})$ rule, which means $\Gamma \in \bigcap_{\psi \in \Gamma} [\![\psi]\!]^{\mathfrak{M}_{\mathcal{N}}} = [\![\Gamma]\!]^{\mathfrak{M}_{\mathcal{N}}}$, in light of Lemma 15. By soundness of lattice logic in Proposition 8 for the assumption $\Gamma \vdash_{\mathcal{L}} \varphi$ we also have $\Gamma \in [\![\varphi]\!]^{\mathfrak{M}_{\mathcal{N}}}$, which when combined with Lemma 15 once again gives us $\Gamma \in \mathsf{Nf}(\varphi)$ and thus $\Gamma \vdash_{\mathrm{NF}} \varphi$ as desired. The converse direction follows readily by induction on the derivation of the judgement $\Gamma \vdash_{\mathrm{NF}} \varphi$. ◀

2.4 Digression: weak cover systems

What happens if we drop (identity) from our cover systems? Let us call the resulting cover systems and models *weak cover systems* and *models*. We use the same definition of localised subsets and the interpretation from Definition 5, and prove persistence as in Lemma 7. Then it turns out that this seemingly minor semantic modification has a large impact on the logic.

▶ **Example 17.** Consider the cover system $(W, \lhd)$ with $W = \{w, v\}$ and $w \lhd \{v\}$ and $v \lhd \{v\}$. Then the localised subsets are $\emptyset, \{w\}$ and W. The set $\{v\}$ is not localised, because $w \lhd \{v\} \subseteq \{v\}$ and $w \notin \{v\}$. Evaluate proposition letters p and q as $V(p) = \{w\}$ and $V(q) = \emptyset$. Then we have $w \models p$ while $w \not\models p \vee q$, invalidating ($\vee$-IN$_1$).





▶ **Example 18.** Let $W = \{w, v, u, s, t\}$ and define $\lhd$ by $w \lhd \{v, u\}$ and $u \lhd \{s, t\}$. This trivially satisfies (transitivity) because each cover contains a world that does not have any cover. With $V(p) = \{v\}$, $V(q) = \{s\}$ and $V(r) = \{t\}$, it can be shown that $w \models p \vee (q \vee r)$ but $w \not\models (p \vee q) \vee r$, invalidating associativity of $\vee$.

We can obtain a proof system corresponding to this semantics by weakening the disjunction rules from Definition 1 as follows.

▶ **Definition 19.** Let $\mathcal{WL}$, for *weak lattice logic*, be the logic given by the rules (HYP), ($\top$-IN), ($\bot$-EL), ($\wedge$-EL$_1$), ($\wedge$-EL$_2$) and ($\wedge$-IN) from Definition 1, together with:

$$\frac{\Gamma \vdash \varphi \vee \psi \quad \varphi \vdash \chi \quad \psi \vdash \xi}{\Gamma \vdash \chi \vee \xi}\ (\vee\text{-MON}) \qquad \frac{\Gamma \vdash \varphi \vee \varphi}{\Gamma \vdash \varphi}\ (\vee\text{-ID})$$

$$\frac{\Gamma \vdash \varphi \vee \varphi}{\Gamma \vdash \varphi \vee \psi}\ (\vee\text{-WK}) \qquad \frac{\Gamma \vdash \psi \vee \varphi}{\Gamma \vdash \varphi \vee \psi}\ (\vee\text{-COMM})$$

We write $\Gamma \vdash_{\mathcal{WL}} \varphi$ if the judgement $\Gamma \vdash \varphi$ can be derived in $\mathcal{WL}$.

We cannot have contexts in two of the premises of ($\vee$-MON) as this would introduce distributivity. Also note that we can combine ($\vee$-MON) and ($\vee$-ID) to obtain ($\vee$-EL). However, in absence of ($\vee$-IN$_1$) and ($\vee$-IN$_2$) we cannot derive ($\vee$-MON) from ($\vee$-EL).

Write $\Gamma \models_{\mathcal{WL}} \varphi$ if $\Gamma \models^{\mathfrak{M}} \varphi$ for all weak cover models $\mathfrak{M}$. A routine induction on the structure of a proof of $\Gamma \vdash_{\mathcal{WL}} \varphi$ then yields soundness, i.e. $\Gamma \vdash_{\mathcal{WL}} \varphi$ implies $\Gamma \models_{\mathcal{WL}} \varphi$. Completeness follows from a modification of the canonical model construction from Section 2.2.

▶ **Definition 20.** Define the covering relation $\lhd_{\mathcal{WL}} \subseteq \mathsf{Ctx}_{\mathbf{L}} \times \mathcal{P}(\mathsf{Ctx}_{\mathbf{L}})$ inductively by ($\lhd$-ID) and ($\lhd\vee$) from Definition 9, together with

$$\frac{\Gamma \in \mathsf{Ant}(\varphi \vee \psi)}{\Gamma \lhd \{\{\varphi\}, \{\psi\}\}}\ (\lhd_{\mathcal{WL}}\vee)$$

As before, let $V_{\mathcal{WL}}(p) = \mathsf{Ant}(p)$ and define the canonical model as $\mathfrak{M}_{\mathcal{WL}} = (\mathsf{Ctx}_{\mathbf{L}}, \lhd_{\mathcal{WL}}, V_{\mathcal{WL}})$.

This canonical model allows us to prove analogues of Lemmas 10 and 11, so that:

▶ **Theorem 21.** *For any $\Gamma \in \mathsf{Ctx}_{\mathbf{L}}$ and $\varphi \in \mathbf{L}$, we have $\Gamma \models_{\mathcal{WL}} \varphi$ if and only if $\Gamma \vdash_{\mathcal{WL}} \varphi$.*

3 Modal lattice logic

We extend lattice logic with a modal operator $\heartsuit$. The resulting logic is a minimal modal logic that algebraically corresponds to a lattice accompanied by a unary monotone operator $\heartsuit$. Formally, we define the language $\mathbf{M}$ and collection of contexts $\mathsf{Ctx}_{\mathbf{M}}$ via:

$$\varphi ::= p \in \mathrm{Prop} \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \heartsuit\varphi \qquad \Gamma ::= \cdot \mid \Gamma, \varphi.$$

▶ **Definition 22.** Define the proof system $\mathcal{M}$ for modal lattice logic as the extension of the inference rules in Definition 1 for lattice logic with the following "monotonicity" rule:

$$\frac{\Gamma \vdash \heartsuit\varphi \qquad \varphi \vdash \psi}{\Gamma \vdash \heartsuit\psi}\ (\heartsuit\text{-MON})$$

We write $\Gamma \vdash_{\mathcal{M}} \varphi$ to emphasise that sequent $\Gamma \vdash \varphi$ is derivable in this system.

3.1 Modal cover models

Let us now show how to model modal lattice logic using cover semantics. We inerpret the modal operator $\heartsuit$ by extending cover models with an additional cover relation $\blacktriangleleft$.

▶ **Definition 23.** A *modal cover system* is a tuple $(W, \lhd, \blacktriangleleft)$ that consists of a cover system $(W, \lhd)$ and a modal cover relation $\blacktriangleleft \subseteq W \times \mathcal{P}(W)$ such that:

$$\text{if } w \lhd \alpha \text{ and } v \blacktriangleleft \alpha_v \text{ for all } v \in \alpha, \text{ then } w \blacktriangleleft \bigcup_{v\in\alpha} \alpha_v \qquad (\lhd\blacktriangleleft\text{-transitivity})$$

A *modal cover model* $\mathfrak{M} = (W, \lhd, \blacktriangleleft, V)$ consists of a modal cover sysetm $(W, \lhd, \blacktriangleleft)$ and a valuation $V : \mathrm{Prop} \to \mathrm{Loc}(W, \lhd)$. The interpretation of $\mathbf{M}$-formulas in a modal cover model $\mathfrak{M}$ is given by extending the clauses from Definition 5 with

$$\mathfrak{M}, w \models \heartsuit\varphi \quad \text{iff} \quad \text{there exists } \beta \text{ such that } w \blacktriangleleft \beta \subseteq [\![\varphi]\!].$$

The relationship $w \blacktriangleleft \alpha$ may be intuitively understood as asserting that set α is a collection of possible futures or refinement of knowledge that is currently known at world w. The condition ($\lhd\blacktriangleleft$-transitivity) states that refinements α_v of members v in a locality α of a world w are collectively a refinement of w. Formally this condition ensures that the interpretation of every formula is a localised subset of $(W, \lhd)$. In light of this condition, a routine extension of the proof of Proposition 8 yields soundness for the proof system $\mathcal{M}$.

▶ **Proposition 24.** *If $\Gamma \vdash_{\mathcal{M}} \varphi$ then $\Gamma \models^{\mathfrak{M}} \varphi$ for every modal cover model $\mathfrak{M}$.*

For completeness, we extend the construction from Definition 9.

▶ **Definition 25.** Define the cover relation $\lhd_{\mathcal{M}}$ via the clauses ($\lhd$-ID), ($\lhd\bot$) and ($\lhd\vee$) from Definition 9, and generate a modal cover relation $\blacktriangleleft_{\mathcal{M}} \subseteq \mathsf{Ctx}_{\mathbf{M}} \times \mathcal{P}(\mathsf{Ctx}_{\mathbf{M}})$ via:

$$\frac{\Gamma \in \mathsf{Ant}(\heartsuit\varphi)}{\Gamma \blacktriangleleft_{\mathcal{M}} \{\{\varphi\}\}}\ (\blacktriangleleft\heartsuit) \quad \frac{\Gamma \in \mathsf{Ant}(\bot)}{\Gamma \blacktriangleleft_{\mathcal{M}} \emptyset}\ (\blacktriangleleft\bot) \quad \frac{\Gamma \in \mathsf{Ant}(\varphi \vee \psi) \quad \{\varphi\} \blacktriangleleft_{\mathcal{M}} \alpha_1 \quad \{\psi\} \blacktriangleleft_{\mathcal{M}} \alpha_2}{\Gamma \blacktriangleleft_{\mathcal{M}} \alpha_1 \cup \alpha_2}\ (\blacktriangleleft\vee)$$

Define $V_{\mathcal{M}}(p) = \mathsf{Ant}(p)$ and let $\mathfrak{M}_{\mathcal{M}} = (\mathsf{Ctx}_{\mathbf{M}}, \lhd_{\mathcal{M}}, \blacktriangleleft_{\mathcal{M}}, V_{\mathcal{M}})$.

▶ **Lemma 26.** *The tuple $\mathfrak{M}_{\mathcal{M}} = (\mathsf{Ctx}_{\mathbf{M}}, \lhd_{\mathcal{M}}, \blacktriangleleft_{\mathcal{M}}, V_{\mathcal{M}})$ is a modal cover model.*

Proof. It can be shown as in Lemma 10 that $\lhd_{\mathcal{M}}$ satisfies (identity) and (transitivity) and that $V_{\mathcal{M}}(p) \in \mathrm{Loc}(\mathsf{Ctx}_{\mathbf{M}}, \lhd_{\mathcal{M}})$ for each p. For ($\lhd\blacktriangleleft$-transitivity), suppose $\Gamma \lhd_{\mathcal{M}} \alpha$ and $\Delta \blacktriangleleft_{\mathcal{M}} \beta_\Delta$ for all $\Delta \in \alpha$. We prove by induction on the construction of $\Gamma \lhd_{\mathcal{M}} \alpha$ that $\Gamma \blacktriangleleft_{\mathcal{M}} \bigcup_{\Delta\in\alpha} \beta_\Delta$. If it is constructed using ($\lhd$-ID) then we have $\Gamma \blacktriangleleft \bigcup_{\Delta\in\alpha} \beta_\Delta$ by definition, and if it is constructed using ($\lhd\bot$) then $\bigcup_{\Delta\in\alpha} \beta_\Delta = \emptyset$ and we can use ($\blacktriangleleft\bot$). Lastly, if the last rule in the construction of $\Gamma \lhd_{\mathcal{M}} \alpha$ is ($\lhd\vee$), then the desired cover follows from the induction hypothesis and ($\blacktriangleleft\vee$). ◀

▶ **Lemma 27** (Truth lemma). *For all $\Gamma \in \mathsf{Ctx}_{\mathbf{M}}$ and $\varphi \in \mathbf{M}$, we have $\mathfrak{M}_{\mathcal{M}}, \Gamma \models \varphi$ iff $\Gamma \vdash_{\mathcal{M}} \varphi$.*

Proof. We extend the proof of Lemma 11 with one more case, accounting for formulas of the shape $\varphi = \heartsuit\psi$. If $\Gamma \vdash_{\mathcal{M}} \heartsuit\psi$ then by ($\blacktriangleleft\heartsuit$) we have $\Gamma \blacktriangleleft_{\mathcal{M}} \{\{\varphi\}\} \subseteq [\![\varphi]\!]$, so $\mathfrak{M}_{\mathcal{M}}, \Gamma \models \heartsuit\psi$. For the converse, suppose $\mathfrak{M}_{\mathcal{M}}, \Gamma \models \heartsuit\psi$. Then there exists a cover α such that $\Gamma \blacktriangleleft_{\mathcal{M}} \alpha \subseteq [\![\psi]\!]$. We use induction on the construction of $\Gamma \blacktriangleleft_{\mathcal{M}} \alpha$ to show that $\Gamma \vdash_{\mathcal{M}} \heartsuit\psi$.

Case 1. If $\Gamma \blacktriangleleft_{\mathcal{M}} \alpha$ is obtained using ($\blacktriangleleft\heartsuit$) then there exists χ such that $\Gamma \vdash_{\mathcal{M}} \heartsuit\chi$ and $\{\chi\} = \alpha$. The assumption $\alpha \subseteq [\![\psi]\!]$ entails $\{\chi\} \in [\![\psi]\!]$, i.e. $\mathfrak{M}_{\mathcal{M}}, \{\chi\} \models \psi$, so by the induction hypothesis $\chi \vdash \psi$. This allows us to use ($\heartsuit$-MON) to obtain $\Gamma \vdash \heartsuit\psi$, as desired.

Case 2. If $\Gamma \blacktriangleleft_{\mathcal{M}} \alpha$ is obtained using ($\blacktriangleleft\bot$), then $\Gamma \vdash \bot$, so ($\bot$-EL) gives $\Gamma \vdash \heartsuit\psi$.

Case 3. Finally, if $\Gamma \blacktriangleleft_{\mathcal{M}} \alpha$ is obtained using ($\blacktriangleleft\vee$), then there exist formulas χ, ξ and covers α_1, α_2 such that $\{\chi\} \blacktriangleleft_{\mathcal{M}} \alpha_1$ and $\{\xi\} \blacktriangleleft_{\mathcal{M}} \alpha_2$ and $\alpha = \alpha_1 \cup \alpha_2$. Then $\alpha_1 \subseteq [\![\psi]\!]$ and $\alpha_2 \subseteq [\![\psi]\!]$, so by induction we get $\chi \vdash_{\mathcal{M}} \heartsuit\psi$ and $\xi \vdash_{\mathcal{M}} \heartsuit\psi$. Using ($\vee$-EL) then gives $\Gamma \vdash_{\mathcal{M}} \heartsuit\psi$. ◀

▶ **Theorem 28.** *For any $\Gamma \in \mathsf{Ctx}_{\mathbf{M}}$ and $\varphi \in \mathbf{M}$, we have $\Gamma \models_{\mathcal{M}} \varphi$ if and only if $\Gamma \vdash_{\mathcal{M}} \varphi$.*

3.2 Further extensions

The strength of a semantics for a minimal modal logic lies in its ability to accommodate extensions of the logic with additional modal axioms or inference rules. The following table shows how we can accommodate in cover semantics some common modal axioms and rules considered in the modal logic literature. To model a logic with a property on the left, we impose the corresponding condition on the right on a cover system for the logic.

Logical axiom or rule	Sufficient frame condition
$(\Gamma \vdash \heartsuit\bot)/(\Gamma \vdash \bot)$	$w \blacktriangleleft \emptyset$ implies $w \lhd \emptyset$
$(\cdot \vdash \varphi)/(\Gamma \vdash \heartsuit\varphi)$	every world has a modal cover
If $\Gamma \vdash \heartsuit\varphi \wedge \heartsuit\psi$ then $\Gamma \vdash \heartsuit(\varphi \wedge \psi)$	if $w \blacktriangleleft \alpha$ and $w \blacktriangleleft \beta$ then $w \blacktriangleleft \gamma \subseteq \alpha \cap \beta$ for some γ
If $\Gamma \vdash \varphi$ then $\Gamma \vdash \heartsuit\varphi$	$w \blacktriangleleft \{w\}$ for all worlds
If $\Gamma \vdash \heartsuit\heartsuit\varphi$ then $\Gamma \vdash \heartsuit\varphi$	$\blacktriangleleft\blacktriangleleft$-transitivity

Observe that the canonical model $\mathfrak{M}_{\mathcal{M}}$ already has the potential to validate the rule that infers $\Gamma \vdash \bot$ from $\Gamma \vdash \heartsuit\bot$ due to the below lemma. Canonical models for other extensions can be constructed by defining suitable cover relations, akin to [50].

▶ **Lemma 29.** *In presence of $(\Gamma \vdash \heartsuit\bot)/(\Gamma \vdash \bot)$, we have that $\Gamma \blacktriangleleft \emptyset$ implies $\Gamma \lhd \emptyset$ in $\mathfrak{M}_{\mathcal{M}}$.*

Proof. By induction on the construction of $\Gamma \blacktriangleleft_{\mathcal{M}} \emptyset$. It cannot be constructed using ($\blacktriangleleft\heartsuit$). If it is constructed using ($\blacktriangleleft\bot$), then we have $\Gamma \in \mathsf{Ant}(\bot)$, to which we apply the ($\lhd\bot$) rule to obtain $\Gamma \lhd \emptyset$. Lastly, if it is constructed using ($\blacktriangleleft\vee$) as the last rule, then for some formulas φ, ψ, it follows that $\Gamma \in \mathsf{Ant}(\varphi \vee \psi)$ and from the induction hypotheses that $\{\varphi\} \lhd \emptyset$ and $\{\psi\} \lhd \emptyset$, which allows us to apply the ($\lhd\vee$) rule to obtain a construction of $\Gamma \lhd \emptyset$. ◀

4 First-order lattice logic

We extend lattice logic with constants, predicates and quantifiers. Syntactically, we extend the proof system from Definition 1 with the usual quantifier rules. Semantically, this is reflected by extending a cover system with a universe U of names, and an interpretation of the constants and predicates. Completeness is proven constructively using a modification of the completeness for first-order intuitionistic logic [24, Section 8].

4.1 Cover models for first-order lattice logic

Let Pred be a set of predicate symbols, each accompanied by an arity, Cons a set of constants and Var a countably infinite set of variables. The the set of terms is Terms = Cons∪Var, and the first-order language $\mathbf{F} = \mathbf{F}(\text{Pred}, \text{Cons}, \text{Var})$ and first-order contexts $\mathsf{Ctx}_{\mathbf{F}}$ are given by:

$$\varphi ::= P(t_1, \dots, t_n) \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \forall x\varphi \mid \exists x\varphi \qquad\qquad \Gamma ::= \cdot \mid \Gamma, \varphi$$

where P is an n-ary predicate, $t_1, \dots, t_n \in$ Terms, and $x \in$ Var. We write $\varphi(t/x)$ for the result of substituting every free occurrence of x in φ with the term t.

▶ **Definition 30.** The proof system $\mathcal{F} = \mathcal{F}(\text{Pred}, \text{Cons}, \text{Var})$ is given by extending the rules from Definition 1 with the following quantifier rules:

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \forall x\varphi}\ (\forall\text{-IN}) \qquad \frac{\Gamma \vdash \forall x\varphi}{\Gamma \vdash \varphi(t/x)}\ (\forall\text{-EL}) \qquad \frac{\Gamma \vdash \varphi(t/x)}{\Gamma \vdash \exists x\varphi}\ (\exists\text{-IN}) \qquad \frac{\Gamma \vdash \exists x\varphi \qquad \varphi \vdash \psi}{\Gamma \vdash \psi}\ (\exists\text{-EL})$$

where in ($\forall$-IN) x is not free in Γ, and in ($\exists$-EL) x is not free in ψ.

▶ **Lemma 31.** *The modus ponens* (MP) *rule* $(\Gamma \vdash \varphi \quad \varphi \vdash \psi)/(\Gamma \vdash \psi)$ *is derivable.*

Proof. Let Γ be a context and φ, ψ two first-order formulas. Then the derivation

$$\frac{\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \varphi}\ (\vee\text{-IN}_1) \qquad \varphi \vdash \psi \qquad \varphi \vdash \psi}{\Gamma \vdash \psi}\ (\vee\text{-EL})$$

shows that (MP) is derivable. ◀

The first-order language can be interpreted in cover frames with a universe U of names and an interpretation $\mathcal{I}$ of the constants and predicates as follows.

▶ **Definition 32.** A *first-order cover model* is a structure $\mathfrak{M} = (W, \lhd, U, \mathcal{I})$ where $(W, \lhd)$ is a cover system, U is a universe of names and $\mathcal{I}$ is an interpretation that assigns to each constant c an element $\mathcal{I}(c) \in U$, and to each n-ary predicate P a map $\mathcal{I}(P) : U^n \to \text{Loc}(W, \lhd)$. An *assignment of the variables* for $\mathfrak{M}$ is a function $\rho : \text{Var} \to U$. We extend ρ to a map Terms $\to U$ on terms by setting $\rho(c) = \mathcal{I}(c)$ for every constant c. Furthermore, we denote by $\rho[x := u]$ the assignment that sends x to u and acts as ρ on all other variables.

The interpretation of first-order formulas is given by

$$\begin{array}{lll}
\mathfrak{M}, w \models^\rho P(t_1, \dots, t_n) & \text{iff} & w \in \mathcal{I}(P)(\rho(t_1), \dots, \rho(t_n)) \\
\mathfrak{M}, w \models^\rho \top & & \text{always} \\
\mathfrak{M}, w \models^\rho \bot & \text{iff} & w \lhd \emptyset \\
\mathfrak{M}, w \models^\rho \varphi \wedge \psi & \text{iff} & \mathfrak{M}, w \models^\rho \varphi \text{ and } \mathfrak{M}, w \models^\rho \psi \\
\mathfrak{M}, w \models^\rho \varphi \vee \psi & \text{iff} & \text{there exists } \alpha \subseteq W \text{ such that } w \lhd \alpha \subseteq [\![\varphi]\!]^\rho \cup [\![\psi]\!]^\rho \\
\mathfrak{M}, w \models^\rho \forall x\varphi & \text{iff} & \mathfrak{M}, w \models^{\rho[x:=u]} \varphi \text{ for all } u \in U \\
\mathfrak{M}, w \models^\rho \exists x\varphi & \text{iff} & \text{there exists } \alpha \subseteq W \text{ such that } w \lhd \alpha \\
 & & \text{and for all } v \in \alpha \text{ we have } \mathfrak{M}, v \models^{\rho[x:=u]} \varphi \text{ for some } u \in U
\end{array}$$

Here $[\![\varphi]\!]^\rho := \{w \in W \mid \mathfrak{M}, w \models^\rho \varphi\}$ is the *truth set* of φ in $\mathfrak{M}$ with assignment ρ, which is extended to contexts Γ as expected. We write $\Gamma \models^{\mathfrak{M}} \varphi$ if $[\![\Gamma]\!]^\rho \subseteq [\![\varphi]\!]^\rho$ for all assignments ρ. Finally, we write $\Gamma \models_{\mathcal{F}} \varphi$ if $\Gamma \models^{\mathfrak{M}} \varphi$ for all first-order cover models $\mathfrak{M}$ for $\mathcal{F}$.

Note that we have

$$[\![\forall x\varphi]\!]^\rho = \bigcap_{u\in U}[\![\varphi]\!]^{\rho[x:=u]} \quad \text{and} \quad [\![\exists x\varphi]\!]^\rho = \langle\lhd\rangle\Big(\bigcup_{u\in U}[\![\varphi]\!]^{\rho[x:=u]}\Big). \tag{1}$$

▶ **Lemma 33.** *Let $\mathfrak{M} = (W, \lhd, U, \mathcal{I})$ be a first-order cover model and ρ an assignment of the variables. Then $[\![\varphi]\!]^\rho \in \mathrm{Loc}(w, \lhd)$ for all $\varphi \in \mathbf{F}$.*

Proof. We extend the inductive proof of Lemma 7. If $\varphi = P(t_1, \ldots, t_n)$ then the statement holds by definition, and for quantified formulas we combine Equation (1) and Lemma 4. ◀

▶ **Theorem 34** (soundness). *If $\Gamma \vdash_{\mathcal{F}} \varphi$ then $\Gamma \models^{\mathfrak{M}} \varphi$ for every first-order cover model $\mathfrak{M}$.*

4.2 Completeness

We fix a first order language $\mathbf{F} = \mathbf{F}(\mathrm{Pred}, \mathrm{Cons}, \mathrm{Var})$. To construct a canonical model for $\mathbf{F}$, we use an extended language $\mathbf{F}^{ext} = (\mathrm{Pred}, \mathrm{Cons} \cup \mathrm{Var}, \mathrm{Var}')$ in which the elements of Var are added to Cons, and Var' is a fresh set of variables. Then any $\mathbf{F}$-formula is a $\mathbf{F}^{ext}$-sentence.

▶ **Definition 35.** Let $\mathsf{Ctx}_{\mathbf{F}}$ be the set of $\mathbf{F}$-contexts and let $U_{\mathcal{F}} = \mathrm{Cons} \cup \mathrm{Var}$. Define $\lhd_{\mathcal{F}}$ inductively on $\mathsf{Ctx}_{\mathbf{F}}$ via the clauses from Definition 9 together with

$$\frac{\Gamma \in \mathsf{Ant}(\exists x\varphi) \qquad \{\varphi(c/x)\} \lhd_{\mathcal{F}} \beta_c \text{ for all } c \in U_{\mathcal{F}}}{\Gamma \lhd_{\mathcal{F}} \bigcup_{c\in U_{\mathcal{F}}} \beta_c}\ (\lhd\exists)$$

For each $c \in \mathrm{Cons}$ define $\mathcal{I}_{\mathcal{F}}(c) = c$, and for each n-ary predicate P and terms $t_1, \ldots, t_n$, let $\mathcal{I}_{\mathcal{F}}(P)(t_1, \ldots, t_n) := \mathsf{Ant}(P(t_1, \ldots, t_n))$. Finally, set $\mathfrak{M}_{\mathcal{F}} := (\mathsf{Ctx}_{\mathbf{F}}, \lhd_{\mathcal{F}}, U_{\mathcal{F}}, \mathcal{I}_{\mathcal{F}})$.

▶ **Lemma 36.** *The tuple $\mathfrak{M}_{\mathcal{F}} = (\mathsf{Ctx}_{\mathbf{F}}, \lhd_{\mathcal{F}}, U_{\mathcal{F}}, \mathcal{I}_{\mathcal{F}})$ is a first-order cover model for $\mathcal{F}$.*

Recall that every formula φ in $\mathbf{F}$ is a $\mathbf{F}^{ext}$-sentence, so we can interpret φ in $\mathfrak{M}_{\mathcal{F}}$ without giving an assignment of the variables. We obtain:

▶ **Lemma 37** (Truth lemma). *For all $\varphi \in \mathbf{F}$ and $\Gamma \in \mathsf{Ctx}_{\mathbf{F}}$, $\mathfrak{M}_{\mathcal{F}}, \Gamma \models \varphi$ iff $\Gamma \in \mathsf{Ant}(\varphi)$.*

Proof. We extend the inductive proof from Lemma 11 with cases for the quantifier rules.

Case $\varphi = \forall x\psi$**.** Suppose $\mathfrak{M}_{\mathcal{F}}, \Gamma \models \forall x\psi$. Then by definition $\Gamma \in [\![\psi(t/x)]\!]$ for all $t \in U$, so by induction $\Gamma \in \mathsf{Ant}(\psi(t/x))$ for all $t \in U$. Let y be a variable not occurring in Γ or ψ, then $\Gamma \vdash \psi(y/x)$ and by ($\forall$-IN) $\Gamma \vdash \forall y\psi(y/x)$. Combining ($\forall$-EL) and ($\forall$-IN) shows that $\forall y(\psi(y/x)) \vdash \forall x\psi$, so that (MP) yields $\Gamma \in \mathsf{Ant}(\forall x\psi)$.

For the converse, suppose $\Gamma \in \mathsf{Ant}(\forall x\psi)$. Then $\Gamma \vdash \forall x\psi$, hence by ($\forall$-EL) we have $\Gamma \vdash \psi(t/x)$, i.e. $\Gamma \in \mathsf{Ant}(\psi(t/x))$, for all $t \in U$. By induction $\mathfrak{M}_{\mathcal{F}}, \Gamma \models \psi(t/x)$ for all $t \in U$, hence $\Gamma \in \bigcap_{t\in U}[\![\psi(t/x)]\!] = [\![\forall x\psi]\!]$, i.e. $\mathfrak{M}_{\mathcal{F}}, \Gamma \models \forall x\psi$.

Case $\varphi = \exists x\psi$**.** Suppose $\mathfrak{M}_{\mathcal{F}}, \Gamma \models \exists x\psi$. Then by definition there exists a cover α such that $\Gamma \lhd \alpha \subseteq \bigcup_{t\in U_{\mathcal{F}}}[\![\psi(t/x)]\!]$, so by the induction hypothesis $\Gamma \lhd \alpha \subseteq \bigcup_{t\in U_{\mathcal{F}}} \mathsf{Ant}(\psi(t/x))$. By ($\exists$-IN), $\psi(t/x) \vdash \exists x\psi$. This implies $\mathsf{Ant}(\psi(t/x)) \subseteq \mathsf{Ant}(\exists x\psi)$ for all terms t, so that $\Gamma \lhd \alpha \subseteq \mathsf{Ant}(\exists x\psi)$ and by Lemma 36, $\Gamma \in \mathsf{Ant}(\exists x\psi)$.

For the converse, suppose $\Gamma \in \mathsf{Ant}(\exists x\psi)$. Then by definition and the induction hypothesis

$$\Gamma \lhd \bigcup_{t\in U_{\mathcal{F}}}\{\psi(t/x)\} \subseteq \bigcup_{t\in U_{\mathcal{F}}}\mathsf{Ant}(\psi(t/x)) = \bigcup_{t\in U_{\mathcal{F}}}[\![\psi(t/x)]\!]$$

so by definition $\Gamma \models \exists x\psi$, i.e. $\Gamma \in [\![\exists x\psi]\!]$. ◀

Using the same proof as Theorem 21, but appealing to Lemma 37, yields:

▶ **Theorem 38.** *For all $\varphi \in \mathbf{F}$ and $\Gamma \in \mathsf{Ctx}_{\mathbf{F}}$, we have $\Gamma \models_{\mathcal{F}} \varphi$ if and only if $\Gamma \vdash_{\mathcal{F}} \varphi$.*

$$\Gamma, x : A \vdash_{\mathrm{v}} x : A \qquad \frac{\Gamma \vdash_{\mathrm{v}} x : A \qquad (y \text{ not in } \Gamma)}{\Gamma, y : B \vdash_{\mathrm{v}} x : A} \qquad \frac{\Gamma \vdash_{\mathrm{v}} x : A}{\Gamma \vdash x : A} \qquad \Gamma \vdash \mathsf{unit} : 1$$

$$\frac{\Gamma \vdash t : 0}{\Gamma \vdash \mathsf{abort}\, t : A} \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \mathsf{pair}\ t\, u : A \times B} \qquad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathsf{fst}\, t : A} \qquad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathsf{snd}\, t : B}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{inl}\, t : A + B} \qquad \frac{\Gamma \vdash t : B}{\Gamma \vdash \mathsf{inr}\, t : A + B} \qquad \frac{\Gamma \vdash s : A + B \qquad x_1 : A \vdash t_1 : C \qquad x_2 : B \vdash t_2 : C}{\Gamma \vdash \mathsf{match}\ s\,(x_1 \mapsto t_1)\,(x_2 \mapsto t_2) : C}$$

Figure 1 Well-typed terms in $\mathbb{LC}$.

5 Program calculus based on lattice logic

In this section, we define and analyse a typed program calculus $\mathbb{LC}$ (for “lattice calculus”) based on lattice logic by assigning terms to the proof rules from earlier. The language of types is in one-to-one correspondence with formulas and is defined as follows:

$$A ::= \iota \mid 1 \mid 0 \mid A \times A \mid A + A \qquad\qquad \Gamma ::= \cdot \mid \Gamma, x : A \ (x \text{ not in } \Gamma)$$

The type ι denotes an uninterpreted base type, 1 and 0 denote the unit and empty type, respectively, $A \times B$ denotes the product of two types, and $A + B$ denotes the sum of two types. A context Γ is a finite list $x_1 : A_1, \ldots, x_n : A_n$ of unique type assignments to variables.

The term language of $\mathbb{LC}$ consists of variables $(x, y, \ldots)$ and term constructs for the unit type (unit), empty type (abort), product types (pair, fst, snd) and sum types (inl, inr and match). The typing judgements $\Gamma \vdash t : A$ for $\mathbb{LC}$ are defined in Figure 1. Observe the typing rule for the match construct: given a *scrutinee* $\Gamma \vdash s : A + B$, the *branches* $A \vdash t_1 : C$ and $B \vdash t_2 : C$ of a term $\Gamma \vdash \mathsf{match}\ s\,(x_1 \mapsto t_1)\,(x_2 \mapsto t_2) : C$ cannot access variables in Γ. In light of the match construct, sum types in $\mathbb{LC}$ enable a kind of strict choice where the result of a match can only depend (if at all) on the value produced by the scrutinee.

The equational theory for $\mathbb{LC}$ is given in Figure 2. The η rules state how a term of a given type can be expanded to one in canonical form and the β rules state how terms with “detours” can be reduced when an elimination term construct (e.g. fst) is applied to an introduction construct (e.g. pair). The π rules, also known *permutation* rules [48] or *commuting conversions*, state that an elimination construct (specified by the grammar E in Figure 2) can be permuted with the constructs abort and match. We may extend the equational theory in Figure 2 with the rules π_0^I and π_+^I below to admit a stronger η_+ rule ($\eta_+^\dagger$) for sum types (denoted by $+.\eta^\dagger$ in [41]), which states that for any $\Gamma \vdash s : A + B$ and $x : A + B \vdash t : C$ we have $t[s/x] \approx \mathsf{match}\ s\,(x_1 \mapsto t[\mathsf{inl}\, x_1/x])\,(x_2 \mapsto t[\mathsf{inr}\, x_2/x])$, and similarly a stronger η_0 rule ($\eta_0^\dagger$) stating $t[s/x] \approx \mathsf{abort}\, s$ for any $\Gamma \vdash s : 0$ and $x : 0 \vdash t : C$.

$$I[\mathsf{abort}\, t] \approx \mathsf{abort}\, t \qquad (\pi_0^I)$$

$$I[\mathsf{match}\ s\,(x_1 \mapsto t_1)\,(x_2 \mapsto t_2)] \approx \mathsf{match}\ s\,(x_1 \mapsto I[t_1])\,(x_2 \mapsto I[t_2]) \qquad (\pi_+^I)$$

$$\text{where } I[\,] ::= \mathsf{inl}\,[\,] \mid \mathsf{inr}\,[\,]$$

These rules simplify the categorical semantics of $\mathbb{LC}$, as we will see shortly, but are not always desirable in the study of program calculi, since they lead to an explosion in the size of normal forms and complicate the normalisation algorithm significantly [20, 1, 41].

$$\text{If } \Gamma \vdash t : 1 \text{ then } t \approx \mathsf{unit} \tag{η_1}$$
$$\mathsf{fst}\,(\mathsf{pair}\ t\,u) \approx t \tag{$\beta_{\times 1}$}$$
$$\mathsf{snd}\,(\mathsf{pair}\ t\,u) \approx u \tag{$\beta_{\times 2}$}$$
$$\text{If } \Gamma \vdash t : A \times B \text{ then } t \approx \mathsf{pair}\,(\mathsf{fst}\,t)\,(\mathsf{snd}\,u) \tag{$\eta_\times$}$$
$$\text{If } \Gamma \vdash t : 0 \text{ then } t \approx \mathsf{abort}\,t \tag{η_0}$$
$$\text{If } \Gamma \vdash t : A + B \text{ then } t \approx \mathsf{match}\ t\,(x_1 \mapsto \mathsf{inl}\,x_1)\,(x_2 \mapsto \mathsf{inr}\,x_2) \tag{η_+}$$
$$\mathsf{match}\,(\mathsf{inl}\,s)\,(x_1 \mapsto t_1)\,(x_2 \mapsto t_2) \approx t_1[s/x_1] \tag{β_{+1}}$$
$$\mathsf{match}\,(\mathsf{inr}\,s)\,(x_1 \mapsto t_1)\,(x_2 \mapsto t_2) \approx t_2[s/x_2] \tag{β_{+2}}$$
$$E[\mathsf{abort}\,t] \approx \mathsf{abort}\,t \tag{π_0^E}$$
$$E[\mathsf{match}\ s\,(x_1 \mapsto t_1)\,(x_2 \mapsto t_2)] \approx \mathsf{match}\ s\,(x_1 \mapsto E[t_1])\,(x_2 \mapsto E[t_2]) \tag{π_+^E}$$
$$\text{where } E[\ ] ::= \mathsf{abort}\,[\ ] \mid \mathsf{fst}\,[\ ] \mid \mathsf{snd}\,[\ ] \mid \mathsf{match}\,[\ ]\,(x_1 \mapsto t_1)\,(x_2 \mapsto t_2).$$

Figure 2 Equational theory for $\mathbb{LC}$

5.1 Categorical semantics

Our development in this section will rely on familiarity of some elementary category theory. Detailed definitions can be found in Appendix C, alongside a description of our notation. Briefly, recall that a category $\mathcal{C}$ is *cartesian* when it has a final object (an object 1 with morphisms $\mathsf{term}_A : A \to 1$) and products $(A \times B,\ \mathsf{proj}_1,\ \mathsf{proj}_1,\ \langle -, - \rangle)$ for all objects $(A, B, C, \dots)$ in $\mathcal{C}$, *cocartesian* when it has an initial object $(0,\ \mathsf{init}_A : 0 \to A)$ and coproducts $(A + B,\ \mathsf{inj}_1,\ \mathsf{inj}_2,\ [-, -])$, and *bicartesian* when it is both cartesian and cocartesian.

Given a bicartesian category $\mathcal{B}$ equipped with a valuation object V_ι, suppose we interpret types and contexts of $\mathbb{LC}$ as usual by objects in $\mathcal{B}$ ($[\![\iota]\!] = V_\iota$, $[\![1]\!] = 1$, $[\![0]\!] = 0$, etc.) and terms as morphisms in $\mathcal{B}$. The calculus $\mathbb{LC}$ can naturally be shown to be sound and complete for all such bicartesian categories when its equational theory is extended with rules (π_0^I) and (π_+^I) mentioned previously. This is because the extended equational theory admits the rules $(\eta_0^\dagger)$ and $(\eta_+^\dagger)$, the proof of which can be given by induction on the substituted terms in the respective rules. In the remainder of this section, we will identify a more general class of categorical models of $\mathbb{LC}$ that do not demand $(\eta_0^\dagger)$ and $(\eta_+^\dagger)$.

The key tool in our semantics will be a *monad algebra*. Recollect that a *monad* T on $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ accompanied by a family of morphisms $\eta_A : A \to TA$ and $\mu_A : T^2 A \to TA$ subject to the usual monad laws. A T-monad algebra $\mathfrak{A}_A = (A, \ell_A)$ is an object A in $\mathcal{C}$ coupled with an "algebra map" morphism $\ell_A : TA \to A$ in $\mathcal{C}$ subject to certain conditions, and a homomorphism $h : \mathfrak{A}_A \Rightarrow \mathfrak{A}_B$ is a morphism $h : A \to B$ that commutes with the algebra maps. The main idea is that the monad T will generalise the operator $\langle \lhd \rangle$ from earlier while T-monad algebras will generalise localised sets.

▶ **Definition 39.** Let T be a monad on a category $\mathcal{C}$. Let $\mathcal{C}^T$ be the category with T-algebras as objects, and whose morphisms $\mathfrak{A}_A \to \mathfrak{A}_B$ are given by morphisms $A \to B$ in $\mathcal{C}$.

$\mathcal{C}^T$ is *not* the Eilenberg–Moore category, as morphisms in $\mathcal{C}^T$ *need not be homomorphisms.*

▶ **Proposition 40.** *There is a "free" functor $J : \mathcal{C} \to \mathcal{C}^T$ that maps an object A in $\mathcal{C}$ to the T-monad algebra $(TA, \mu_A : T^2 A \to TA)$ and a morphism $f : A \to B$ in $\mathcal{C}$ to the morphism $Tf : TA \to TB$ in $\mathcal{C}$, which is a morphism $(TA, \mu_A) \to (TB, \mu_B)$ in $\mathcal{C}^T$.*

▶ **Lemma 41** (Key lemma for Theorem 42). *Given an arbitrary bicartesian category $\mathcal{B}$ equipped with a monad T, the following hold for the category $\mathcal{B}^T$ determined by Definition 39:*

1. *$\mathcal{B}^T$ is a cartesian category.*
2. *$J0$ is a weak initial object in $\mathcal{B}^T$ (i.e. $\mathsf{init}_{\mathfrak{A}_A} : J0 \to \mathfrak{A}_A$ need not be unique) s.t. for any homomorphism $h : \mathfrak{A}_A \Rightarrow \mathfrak{A}_Z$ we have $\mathsf{id}_{J0} = \mathsf{init}_{J0}$ and $h \circ \mathsf{init}_{\mathfrak{A}_A} = \mathsf{init}_{\mathfrak{A}_Z} : J0 \to \mathfrak{A}_Z$.*
3. *$J(A+B)$ is a weak coproduct of the T-algebras $\mathfrak{A}_A$ and $\mathfrak{A}_B$ in $\mathcal{B}^T$ s.t. for any homomorphism $h : \mathfrak{A}_C \Rightarrow \mathfrak{A}_D$ and morphisms $f : \mathfrak{A}_A \to \mathfrak{A}_C$, $g : \mathfrak{A}_B \to \mathfrak{A}_C$ in $\mathcal{B}^T$, we have $\mathsf{id}_{J(A+B)} = [\mathsf{inj}_1, \mathsf{inj}_2]$ and $h \circ [f, g] = [h \circ f, h \circ g] : J(A+B) \to \mathfrak{A}_C$.*
4. *Given any T-algebras $\mathfrak{A}_A$, $\mathfrak{A}_B$, $\mathfrak{A}_Z$, the following morphisms in $\mathcal{B}^T$ are homomorphisms:*
   a. $\mathsf{proj}_1 : \mathfrak{A}_A \times \mathfrak{A}_B \to \mathfrak{A}_A$ *and* $\mathsf{proj}_2 : \mathfrak{A}_A \times \mathfrak{A}_B \to \mathfrak{A}_B$
   b. $\mathsf{init}_{\mathfrak{A}_A} : J0 \to \mathfrak{A}_A$ *and* $[f, g] : J(A+B) \to \mathfrak{A}_Z$ *for any* $f : \mathfrak{A}_A \to \mathfrak{A}_Z$, $g : \mathfrak{A}_B \to \mathfrak{A}_Z$

Given a monad T on a bicartesian category $\mathcal{B}$, let us refer to the category of algebras $\mathcal{B}^T$ as a *categorical model* of $\mathbb{LC}$ when it is equipped with some T-monad algebra V_ι for valuation of ι. We can interpret types and contexts in any categorical model of $\mathbb{LC}$ as follows:

$$\begin{array}{lll} [\![\iota]\!] = V_\iota & & \\ [\![1]\!] = 1 & [\![0]\!] = J(0) & [\![\cdot]\!] = 1 \\ [\![A \times B]\!] = [\![A]\!] \times [\![B]\!] & [\![A + B]\!] = J([\![A]\!] + [\![B]\!]) & [\![\Gamma, x : A]\!] = [\![\Gamma]\!] \times [\![A]\!] \end{array}$$

The product of two interpetations is given by the cartesian structure of $\mathcal{B}^T$ due to Lemma 41(1). While $J(0)$ is not an initial object in $\mathcal{B}^T$, it forms a sufficiently weak initial object, as stated in Lemma 41(2), that validates the expansion rule η_0 and permutation rules π_0^E in Figure 2. Similarly, $J([\![A]\!] + [\![B]\!])$ forms a sufficiently weak coproduct, as stated in Lemma 41(3), of the interpretations $[\![A]\!]$ and $[\![B]\!]$ that validates the β_{+1}, β_{+2}, η_+ and π_+^E rules in $\mathbb{LC}$.

▶ **Theorem 42** (Soundness of categorical semantics). *For every term $\Gamma \vdash t : A$ in $\mathbb{LC}$ there is a morphism $[\![t]\!] : [\![\Gamma]\!] \to [\![A]\!]$ in an arbitrary categorical model $\mathcal{C}$ of $\mathbb{LC}$ such that for any other term $\Gamma \vdash u : A$, if $t \approx u$ in the equational theory of $\mathbb{LC}$ (given in Figure 2) then $[\![t]\!] \approx [\![u]\!]$ in $\mathcal{C}$.*

5.2 Denotational semantics using cover models

While it was sufficient to interpret a formula φ as a subset $[\![\varphi]\!] \subseteq W$ for some given cover system $(W, \lhd)$ previously, we need a refined interpretation for types. In this subsection we develop cover semantics for $\mathbb{LC}$ by interpreting types as *families of sets* indexed by W. Recall that a family of sets X indexed by W is a collection of sets $\{X_w\}_{w \in W}$ that consists of a set X_w for each $w \in W$. Write $\mathcal{F}(W)$ for the collection of all families of sets indexed by W and $X \dot{\to} Y$ for the family of functions $\{X_w \to Y_w\}_{w \in W}$ for $X, Y \in \mathcal{F}(W)$. We will replace the cover relation $\lhd$ in a cover system with a *function* Cov that maps a world w and a family of sets X to a set $\mathsf{Cov}(w, X)$ known as the *covering set.* Observe that the function Cov determines a family $\mathsf{Cov}(-, X) = \{\mathsf{Cov}(w, X)\}_{w \in W}$ known as the *covering family* of X.

▶ **Definition 43.** A *refined cover system* for $\mathbb{LC}$ is a tuple (W, Cov) consisting of a set W and a function Cov that defines a covering set $\mathsf{Cov}(w, X)$ for every w and X, equipped with the following families of functions that satisfy the subsequent coherence conditions:

- $\mathsf{map}\langle f \rangle : \mathsf{Cov}(-, X) \dot{\to} \mathsf{Cov}(-, Y)$ for any $f : X \dot{\to} Y$
- $\mathsf{iden}_X : X \dot{\to} \mathsf{Cov}(-, X)$ for any $X \in \mathcal{F}(W)$
- $\mathsf{trans}\langle f \rangle : \mathsf{Cov}(-, X) \dot{\to} \mathsf{Cov}(-, Y)$ for any $f : X \dot{\to} \mathsf{Cov}(-, Y)$
- $\mathsf{map}\langle \mathsf{id} \rangle = \mathsf{id}$ and $\mathsf{map}\langle f \rangle \circ \mathsf{map}\langle g \rangle = \mathsf{map}\langle f \circ g \rangle$
- $\mathsf{map}\langle f \rangle \circ \mathsf{iden} = \mathsf{iden} \circ f$ and $\mathsf{map}\langle f \rangle \circ \mathsf{trans}\langle g \rangle = \mathsf{trans}\langle \mathsf{map}\langle f \rangle \circ g \rangle$
- $\mathsf{trans}\langle \mathsf{trans}\langle f \rangle \circ g \rangle = \mathsf{trans}\langle f \rangle \circ \mathsf{trans}\langle g \rangle$ and $\mathsf{trans}\langle \mathsf{iden} \rangle = \mathsf{id}$ and $\mathsf{trans}\langle f \rangle \circ \mathsf{iden} = f$

The first of the three coherence conditions says that $\mathsf{map}\langle - \rangle$ preserves identity ($\mathsf{id} : X \dot{\to} X$) and composition of families of functions, the second states iden and $\mathsf{trans}\langle - \rangle$ are "natural", i.e. commute with $\mathsf{map}\langle - \rangle$, and third states that $\mathsf{trans}\langle - \rangle$ is associative and iden is its unit.

▶ **Definition 44.** We say that $X \in \mathcal{F}(W)$ is *localised family* when there exists a family of functions $\ell_X : \mathsf{Cov}(-, X) \dot{\to} X$ such that $\ell_X \circ \mathsf{iden}_X = \mathsf{id}$ and $\ell_X \circ \mathsf{map}\langle \ell_X \rangle = \ell_X \circ \mathsf{trans}\langle \mathsf{id} \rangle$. Let us write $\mathrm{LocFam}(W, \mathsf{Cov})$ for the collection of localised families determined by (W, Cov).

▶ **Proposition 45.** *Given a set W, families of sets indexed by W form a bicartesian category equipped with a monad Cov that maps a family X to its covering family $\mathsf{Cov}(-, X)$. Further, a localised family $X \in \mathrm{LocFam}(W, \mathsf{Cov})$ is a Cov-monad algebra with ℓ_X as the algebra map.*

A *refined cover model* for $\mathbb{LC}$ is a triple (W, Cov, V) with a cover system (W, Cov) and a valuation V that assigns a localised family $V(\iota)$ to the base type ι. By virtue of the soundness of categorical semantics for $\mathbb{LC}$ in Theorem 42, Proposition 45 gives us that the category of Cov-algebras $\mathcal{F}(W)^{\mathsf{Cov}}$ is a categorical model of $\mathbb{LC}$. We thus obtain an equationally sound interpretation of types A and contexts Γ in $\mathbb{LC}$ as localised families $[\![A]\!], [\![\Gamma]\!] \in \mathrm{LocFam}(W, \mathsf{Cov})$ and terms $\Gamma \vdash t : A$ in $\mathbb{LC}$ as families of functions $[\![t]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$.

▶ **Corollary 46** (Denotational semantics for $\mathbb{LC}$ via cover models)**.** *Given an arbitrary cover model of $\mathbb{LC}$, if $t \approx u$ in $\mathbb{LC}$ then the families of functions $[\![t]\!]$ and $[\![u]\!]$ are equivalent.*

For completeness of proof terms we can once again construct a canonical model. Define a family of normal forms $\mathsf{Nf}(\Gamma, A) = \{n \mid \Gamma \vdash_{\mathrm{NF}} n : A\}_{\Gamma \in \mathsf{Ctx}}$, akin to the set of normal antecedents in Section 2.3, and similarly $\mathsf{Ne}(\Gamma, A) = \{n \mid \Gamma \vdash_{\mathrm{NE}} n : A\}_{\Gamma \in \mathsf{Ctx}}$ as the family of neutrals. Also define a canonical model $\mathfrak{M}_{\mathcal{N}} = (\mathsf{Ctx}, \mathsf{Cov}_{\mathcal{N}}, \mathsf{Nf}(-, \iota))$, where $\mathsf{Cov}_{\mathcal{N}}$ is given for any $X \in \mathcal{F}(\mathsf{Ctx})$ and context Γ by the covering set $\mathsf{Cov}_{\mathcal{N}}(\Gamma, X)$ defined inductively as:

- If we have $x \in X_\Gamma$, then infer $\mathsf{leaf}(x) \in \mathsf{Cov}_{\mathcal{N}}(\Gamma, X)$
- If we have $x \in \mathsf{Ne}(\Gamma, 0)$, then infer $\mathsf{dead}(x) \in \mathsf{Cov}_{\mathcal{N}}(\Gamma, X)$
- If we have $x \in \mathsf{Ne}(\Gamma, A + B)$, $m_1 \in \mathsf{Cov}_{\mathcal{N}}(\{A\}, X)$ and $m_2 \in \mathsf{Cov}_{\mathcal{N}}(\{B\}, X)$, then infer $\mathsf{branch}(x, m_1, m_2) \in \mathsf{Cov}_{\mathcal{N}}(\Gamma, X)$

The tags leaf, dead and branch are "constructors" in the sense of type theory that record the applied construction rule within the elements of a covering set. As before, we have:

▶ **Theorem 47** (Completeness of normal forms in $\mathbb{LC}$)**.** *Given a term $\Gamma \vdash t : A$, or a family of functions $[\![\Gamma]\!] \dot{\to} [\![A]\!]$ for all cover models of $\mathbb{LC}$, we can construct a normal form $\Gamma \vdash_{\mathrm{NF}} n : A$.*

6 Conclusion and future work

We have investigated cover semantics for lattice logics. Using Goldblatt's work as a starting point, we used a simpler notion of cover systems and provided constructive soundness and completeness proofs for lattice logic as well as modal, first-order and program calculus extensions. This opens the door to myriad directions for future research.

First, it would be interesting to further develop the model theory of cover semantics. This could include a notion of morphism between cover systems akin to [44, Definition 2.7], developing generated subframes and disjoint unions as in [8, Section 2.1], and developing a constructive categorical duality for (modal) lattices.

A second natural direction for further research is to capture more modal lattice logics. There are various approaches towards extending lattice logic with modal operators, including extensions with independent [13] and interacting [7] $\Box$- and $\Diamond$-modalities, temporal extensions [32], and non-distributive description logics [2]. It would be interesting to see if these can all be studied using covers, thus using cover semantics as a universal modal framework.

Next, the embedding of lattice logic into classical monotone S4 from Remark 13 warrants further study. It can be viewed as an analogue of the Gödel-McKinsey-Tarski translation of intuitionistic logic into classical normal $\mathcal{S}4$ [22, 42, 15], which gives rise to the famous Blok-Esakia theorem [9, 16, 17]. Does this theorem have an analogue in the lattice logic setting?

Finally, it would be interesting to investigate the connection between modal and first-order lattice logic. A similar approach was used to obtain intuitionistic modal logics from classical ones [47, 30]. In the setting of lattice logic, the usual standard translation [8, Section 2.4], using the diamond-clause for $\heartsuit$, yields a translation of $\mathbf{M}$ into a the first-order language with one binary predicate and a unary predicate for each proposition letter. What modal lattice logic is such that derivability of $\Gamma \vdash \varphi$ coincides with derivability of $\mathrm{st}_x(\Gamma) \vdash \mathrm{st}_x(\varphi)$?

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A Embedding lattice logic into classical monotone modal logic

We provide details for Remark 13. Let $\mathbf{C}$ be a classical modal language with one modal operator $\bigcirc$. Define the logic $\mathcal{MS}4$ as the extension of classical monotone modal logic with the axioms $p \to \bigcirc p$ and $\bigcirc\bigcirc p \to \bigcirc p$. Since cover systems correspond bijectively to neighbourhood frames satisfying counterparts of (identity) and (transitivity), we can transfer the usual neighbourhood semantics for monotone modal logic to models based on cover systems.

Concretely, by a *classical cover model* we mean a tuple $\mathfrak{C} = (W, \lhd, V)$ consisting of a cover system $(W, \lhd)$ and a valuation V that assigns to each proposition letter p a (not necessarily localised) subset $V(p)$ of W. The classical connectives are interpreted as usual, and the modal operator is interpreted by

$$w \models^{cl} \bigcirc\varphi \quad \text{iff} \quad w \lhd \alpha \subseteq [\![\varphi]\!] \text{ for some cover } \alpha.$$

We write $\models^{cl}$ to highlight that we are interpreting classical modal formulas in a classical way. In particular, disjunction is interpreted locally, i.e. $w \models^{cl} \varphi \vee \psi$ if $w \models^{cl} \varphi$ or $w \models^{cl} \psi$ (or both). Besides, note that if the cover relation is upward closed, i.e. if $w \lhd \alpha \subseteq \beta$ implies $w \lhd \beta$, then the interpretation simplifies to $w \models^{cl} \bigcirc\varphi$ iff $w \lhd [\![\varphi]\!]$.

We can use existing results for classical monotone modal logic form [31] to establish that the logic $\mathcal{MS}4$ is sound and complete with respect to the class of cover systems.

▶ **Theorem 48.** *The logic $\mathcal{MS}4$ is sound and complete with respect to the class of classical cover models.*

Proof. Soundness follows from the tact that (identity) and (transitivity) can be used to prove validity the additional axioms (with respect to all valuations, not just the localised ones). For completeness, we use the fact that the axioms $p \to \bigcirc p$ and $\bigcirc\bigcirc p \to \bigcirc p$ are equivalent to dual KW-formulas [31, Definition 5.13]. Therefore [31, Theorem 10.44] entails that $\mathcal{MS}4$ is sound and complete with respect to the class of monotone neighbourhood models validating $p \to \bigcirc p$ and $\bigcirc\bigcirc p \to \bigcirc p$. Since the monotone neighbourhood frames from [31] can be viewed as special cases of cover systems (namely those in which $w \lhd \alpha \subseteq \beta$ implies $w \lhd \beta$), and on monotone neighbourhood frames (identity) and (transitivity) precisely correspond to the two axioms, it follows that such monotone neighbourhood frames (viewed as cover systems) are contained in the class of all cover systems validating $\mathcal{MS}4$. This yields completeness. ◀

▶ **Lemma 49.** *Let $\mathfrak{C} = (W, \lhd, V)$ be a classical cover model. Define $V^{\lhd}$ by $V^{\lhd}(p) = \langle\lhd\rangle(V(p))$ and let $\mathfrak{C}^{\lhd} = (W, \lhd, V^{\lhd})$. Then $\mathfrak{C}^{\lhd}$ is a cover model for lattice logic, and for all $\varphi \in \mathbf{L}$ and $\Gamma \in \mathsf{Ctx}_{\mathbf{L}}$ we have:*

$$\mathfrak{C}, w \models^{cl} t(\varphi) \quad \textit{iff} \quad \mathfrak{C}^{\lhd}, w \models \varphi. \tag{2}$$

Proof. By Lemma 4(2) $V^{\lhd}(p)$ is localised for each $p \in \mathrm{Prop}$, hence $\mathfrak{C}^{\lhd}$ is a cover model for lattice logic. We prove Equation (2) by induction on φ. The cases for $\varphi = \top$ and $\varphi = \psi \wedge \chi$ proceed as usual. If $\varphi = p \in \mathrm{Prop}$ then we have $t(\varphi) = \bigcirc p$ and

$$\mathfrak{C}, w \models^{cl} t(p) \quad \text{iff} \quad \mathfrak{C}, w \models^{cl} \bigcirc p \quad \text{iff} \quad w \in \langle\lhd\rangle V(p) \quad \text{iff} \quad w \in V^{\lhd}(p) \quad \text{iff} \quad \mathfrak{C}^{\lhd}, w \models p.$$

The case for $\varphi = \bot$ is similar. If $\varphi = \psi \vee \chi$ then we have $t(\varphi) = \bigcirc(t(\psi) \vee t(\chi))$ and we find

$$\begin{aligned} \mathfrak{C}, w \models^{cl} t(\varphi) \quad &\text{iff} \quad \mathfrak{C}, w \models^{cl} \bigcirc(t(\psi) \vee t(\chi)) \\ &\text{iff} \quad w \in \langle\lhd\rangle(\llbracket t(\psi)\rrbracket^{\mathfrak{C}} \cup \llbracket t(\chi)\rrbracket^{\mathfrak{C}}) \\ &\text{iff} \quad w \in \langle\lhd\rangle(\llbracket\psi\rrbracket^{\mathfrak{C}^{\lhd}} \cup \llbracket\chi\rrbracket^{\mathfrak{C}^{\lhd}}) \qquad \text{(induction hypothesis)} \\ &\text{iff} \quad \mathfrak{C}^{\lhd}, w \models \psi \vee \chi \end{aligned}$$

◀

▶ **Theorem 50.** *For all $\varphi \in \mathbf{L}$ and $\Gamma \in \mathsf{Ctx}_{\mathbf{L}}$, we have $\Gamma \vdash \varphi$ iff $t(\Gamma) \vdash_{\mathcal{MS}4} t(\varphi)$.*

Proof. Suppose $\Gamma \vdash \varphi$. Then $\Gamma \models \varphi$ on the class of cover models for lattice logic. Let $\mathfrak{C}$ be an arbitrary classical cover model. Then by Lemma 49 we obtain $t(\Gamma) \models^{\mathcal{MS}4} t(\varphi)$. This proves that $t(\Gamma) \models t(\varphi)$ on the class of classical cover models, hence by completeness $t(\Gamma) \vdash_{\mathcal{MS}4} t(\varphi)$.

For the converse, suppose $t(\Gamma) \vdash_{\mathcal{MS}4} t(\varphi)$. Then by soundness $t(\Gamma) \models_{\mathcal{MS}4} t(\varphi)$ on the class of classical cover models. Let $\mathfrak{M} = (W, \lhd, V)$ be an arbitrary cover model for lattice logic. Then it is also a classical cover model, and we have $V^{\lhd} = V$. Using Lemma 49 again yields $\Gamma \models^{\mathfrak{M}} \varphi$. Completeness of lattice logic then entails $\Gamma \vdash \varphi$. ◀

B Proofs omitted in Sections 2–4

Proof of Lemma 10. Identity follows immediately from ($\lhd$-ID). For transitivity, suppose $\Gamma \lhd_{\mathcal{L}} \alpha$ and $\Delta \lhd_{\mathcal{L}} \beta_\Delta$ for each $\Delta \in \alpha$. We prove by induction on the construction of α that $\Gamma \lhd_{\mathcal{L}} \bigcup_{\Delta\in\alpha} \beta_\Delta$. If α is constructed using ($\lhd$-ID) then $\alpha = \{\Gamma\}$ and the statement is clear, and if α is constructed using ($\lhd\bot$) then $\alpha = \emptyset$, so that trivially $\Gamma \lhd \emptyset = \bigcup_{\Delta\in\alpha} \beta_\Delta$. So suppose $\Gamma \in \mathsf{Ant}(\varphi \vee \psi)$ and $\{\varphi\} \lhd_{\mathcal{L}} \alpha_1$ and $\{\psi\} \lhd_{\mathcal{L}} \alpha_2$. Then by induction $\{\varphi\} \lhd_{\mathcal{L}} \bigcup_{\Delta\in\alpha_1} \beta_\Delta$ and $\{\psi\} \lhd_{\mathcal{L}} \bigcup_{\Delta\in\alpha_2} \beta_\Delta$, so that ($\lhd\vee$) yields $\Gamma \lhd_{\mathcal{L}} \big(\bigcup_{\Delta\in\alpha_1} \beta_\Delta\big) \cup \big(\bigcup_{\Delta\in\alpha_2} \beta_\Delta\big) = \bigcup_{\Delta\in\alpha} \beta_\Delta$.

To see that $\mathsf{Ant}(\varphi) \in \mathrm{Loc}(\mathsf{Ctx}_{\mathbf{L}}, \lhd_{\mathcal{L}})$, suppose $\Gamma \lhd_{\mathcal{L}} \alpha \subseteq \mathsf{Ant}(\varphi)$. We show by induction on a proof of $\Gamma \lhd_{\mathcal{L}} \alpha$ that $\Gamma \in \mathsf{Ant}(\varphi)$. If $\Gamma \lhd_{\mathcal{L}} \alpha$ is obtained from ($\lhd$-ID) then the assumption gives $\Gamma \in \mathsf{Ant}(\varphi)$, and if it is obtained via ($\lhd\bot$) then Γ proves everything, including φ. Finally, suppose $\Gamma \in \mathsf{Ant}(\psi \vee \chi)$, $\{\psi\} \lhd_{\mathcal{L}} \alpha_1$, $\{\chi\} \lhd_{\mathcal{L}} \alpha_2$ and $\alpha = \alpha_1 \cup \alpha_2$. Then by the induction hypothesis we have $\{\psi\} \in \mathsf{Ant}(\varphi)$ and $\{\chi\} \in \mathsf{Ant}(\varphi)$, so $\psi \vdash \varphi$ and $\chi \vdash \varphi$. It then follows from ($\vee$-EL) that $\Gamma \vdash \varphi$, i.e. $\Gamma \in \mathsf{Ant}(\varphi)$, as desired. ◀

Proof of Theorem 34. Fix an arbitrary first-order cover model $\mathfrak{M}$. We use induction on the length of a derivation for $\Gamma \vdash_{\mathcal{F}} \varphi$ to show that $\llbracket\Gamma\rrbracket^\rho \subseteq \llbracket\varphi\rrbracket^\rho$ for all assignments ρ. If the last rule is one of those from Definition 1, then we proceed as in Proposition 8.

Suppose last rule in the derivation is ($\forall$-IN). Then by assumption $\llbracket\Gamma\rrbracket^\rho \subseteq \llbracket\varphi\rrbracket^\rho$ for every assignment ρ. Since x is not free in Γ, we have $\llbracket\Gamma\rrbracket^\rho \subseteq \llbracket\varphi\rrbracket^{\rho[x:=u]}$ for all $u \in U$. This implies $\llbracket\Gamma\rrbracket \subseteq \bigcap_{u\in U} \llbracket\varphi\rrbracket^{\rho[x:=u]} = \llbracket\forall x\varphi\rrbracket^\rho$. If the last rule is ($\forall$-EL), then by assumption $\llbracket\Gamma\rrbracket^\rho \subseteq \llbracket\forall x\varphi\rrbracket^\rho = \bigcap_{u\in U} \llbracket\varphi\rrbracket^{\rho[x:=u]}$, hence $\llbracket\Gamma\rrbracket^\rho \subseteq \llbracket\varphi\rrbracket^{\rho[x:=u]} = \llbracket\varphi(t/x)\rrbracket^\rho$, for any ρ.

Now suppose last rule in the derivation is ($\exists$-IN). Since $\llbracket\varphi(t/x)\rrbracket^\rho \subseteq \bigcup_{u\in U} \llbracket\varphi\rrbracket^{\rho[x:=u]} = \llbracket\exists x\varphi\rrbracket^\rho$ for any assignment ρ, it follows that $\llbracket\Gamma\rrbracket^\rho \subseteq \llbracket\varphi(t/x)\rrbracket^\rho$ implies $\llbracket\Gamma\rrbracket^\rho \subseteq \llbracket\exists x\varphi\rrbracket^\rho$. Finally, suppose the last rule is ($\exists$-EL). Then by assumption $\llbracket\Gamma\rrbracket^\rho \subseteq \llbracket\exists x\varphi\rrbracket^\rho$ and $\llbracket\varphi\rrbracket^\rho \subseteq \llbracket\psi\rrbracket^\rho$ for every assignment ρ. Since x does not occur free in ψ, we have $\llbracket\varphi\rrbracket^{\rho[x:=u]} \subseteq \llbracket\psi\rrbracket^{\rho[x:=u]} = \llbracket\psi\rrbracket^\rho$ for any assignment ρ, hence

$$\llbracket\exists x\varphi\rrbracket^\rho = \langle\lhd\rangle\Big(\bigcup_{u\in U} \llbracket\varphi\rrbracket^{\rho[x:=u]}\Big) \subseteq \langle\lhd\rangle(\llbracket\psi\rrbracket^\rho) = \llbracket\psi\rrbracket^\rho,$$

as desired. ◀

Proof of Lemma 36. We need to verify that $(\mathsf{Ctx}_{\mathbf{F}}, \lhd_{\mathcal{F}})$ is a cover frame and that $\mathcal{I}_{\mathcal{F}}(P)(t_1, \ldots, t_n)$ is a localised subset of $(\mathsf{Ctx}_{\mathbf{F}}, \lhd_{\mathcal{F}})$ for every n-ary predicate P and terms $t_1, \ldots, t_n$.

$(\mathsf{Ctx}_{\mathbf{F}}, \lhd_{\mathcal{F}})$ *is a cover frame.* Reflexivity of $\lhd_{\mathcal{F}}$ follows immediately from the definition of $\lhd_{\mathcal{F}}$. For transitivity, suppose $\Gamma \lhd_{\mathcal{F}} \alpha$ and $\Delta \lhd_{\mathcal{F}} \beta\Delta$ for each $\Delta \in \alpha$. We prove by induction on the construction of α that $\Gamma \lhd_{\mathcal{F}} \bigcup_{\Delta\in\alpha} \alpha_\Delta$. The cases for ($\lhd$-ID), ($\lhd\bot$) and ($\lhd\vee$) proceed as in Lemma 10. For the additional case of ($\lhd\exists$), suppose $\Gamma \in \mathsf{Ant}(\exists x\varphi)$ and $\{\varphi(c/x)\} \lhd_{\mathcal{F}} \alpha_c$ for all $c \in U_{\mathcal{F}}$ and $\alpha = \bigcup_{c\in U_{\mathcal{F}}} \alpha_c$. Then by assumption $\{\varphi(c/x)\} \lhd_{\mathcal{F}} \bigcup_{\Delta\in\alpha_c} \beta_\Delta$, so ($\lhd\exists$) gives

$$\Gamma \lhd_{\mathcal{F}} \bigcup_{c\in U_{\mathcal{F}}} \Big(\bigcup_{\Delta\in\alpha_c} \beta_\Delta \Big) = \bigcup_{\Delta\in\alpha} \beta_\Delta,$$

as desired.

$\mathcal{I}_{\mathcal{F}}(P)(t_1, \ldots, t_n)$ *is localised.* Suppose $\Gamma \lhd \alpha \subseteq \mathsf{Ant}(P(t_1, \ldots, t_n))$. We use induction on the construction of α to show that $\Gamma \in \mathsf{Ant}(P(t_1, \ldots, t_n))$. If α is constructed using ($\lhd$-ID) or ($\lhd\bot$) then this is obvious. If α is constructed using ($\lhd\vee$) then there exists $\varphi, \psi \in \mathbf{F}$ and $\alpha_1, \alpha_2 \subseteq \mathsf{Ctx}_{\mathbf{F}}$ such that $\Gamma \in \mathsf{Ant}(\varphi \vee \psi)$ and $\{\varphi\} \lhd_{\mathcal{F}} \alpha_1$ and $\{\psi\} \lhd_{\mathcal{F}} \alpha_2$ and $\alpha = \alpha_1 \cup \alpha_2$. Then by induction $\psi \in \mathsf{Ant}(P(t_1, \ldots, t_n))$ and $\psi \in \mathsf{Ant}(P(t_1, \ldots, t_n))$, so that ($\vee$-EL) entails $\Gamma \in \mathsf{Ant}(P(t_1, \ldots, t_n))$.

Finally, suppose α is constructed using ($\lhd\exists$). Then there exists a formula φ and sets $\beta_c \subseteq \mathsf{Ctx}_{\mathbf{F}}$ such that $\Gamma \in \mathsf{Ant}(\exists x\varphi)$ and $\{\varphi(c/x)\} \lhd_{\mathcal{F}} \beta_c$ for each $c \in U_{\mathcal{F}}$ and $\alpha = \bigcup_{c\in U_{\mathcal{F}}} \beta_c$. This implies $\varphi(c/x) \in \mathsf{Ant}(P(t_1, \ldots, t_n))$ for all $c \in U_{\mathcal{F}}$. Pick a variable y that does not appear in φ and is not equal to any of the t_i. Then we have $\varphi(y/x) \vdash P(t_1, \ldots, t_n)$, hence by ($\forall$-IN) we find $\varphi(y/x) \vdash \forall x P(t_1, \ldots, t_n)$. Since x is not free in $\forall x P(t_1, \ldots, t_n)$, we can use ($\exists$-EL) to deduce $\Gamma \vdash \forall x P(t_1, \ldots, t_n)$, hence by ($\forall$-IN) we get $\Gamma \vdash P(t_1, \ldots, t_n)$, i.e. $\Gamma \in \mathsf{Ant}(P(t_1, \ldots, t_n))$, as desired. ◀

C Categorical preliminaries and some proofs omitted in Section 5

Given a category $\mathcal{C}$, an object 1 is a *final object* in $\mathcal{C}$ when it is accompanied by a family of unique morphisms $\mathsf{term}_A : A \to 1$, for objects A in $\mathcal{C}$. Uniqueness means for any morphism $h : A \to 1$ in $\mathcal{C}$, $h = \mathsf{term}_A$. An object 0 is an *initial object* in $\mathcal{C}$ when it is accompanied by a family of unique morphisms $\mathsf{init}_A : 0 \to A$. A *weak initial object* is an object 0 accompanied by a family of morphisms $\mathsf{init}_A : 0 \to A$ that need not be unique.

A *product* of two objects A and B in a category $\mathcal{C}$ is an object $A \times B$ accompanied by "projection" morphisms $\mathsf{proj}_1 : A \times B \to A$ and $\mathsf{proj}_2 : A \times B \to B$, and a "pairing" morphism $\langle f, g\rangle : Z \to A \times B$ for any object Z with morphisms $f : Z \to A$ and $g : Z \to B$ in $\mathcal{C}$. These morphisms are subject to the conditions that:

- $\mathsf{proj}_1 \circ \langle f, g\rangle = f$ and $\mathsf{proj}_2 \circ \langle f, g\rangle = g$
- for any $h : Z \to A \times B$, if $\mathsf{proj}_1 \circ h = f$ and $\mathsf{proj}_2 \circ h = g$ then $h = \langle f, g\rangle$.

Dually, a *coproduct* of two objects A and B is an object $A + B$ accompanied by "injection" morphisms $\mathsf{inj}_1 : A \to A + B$ and $\mathsf{inj}_2 : B \to A + B$, and a "copairing" morphism $[f, g] : A + B \to Z$ for any morphisms $f : A \to Z$ and $g : B \to Z$, subject to the conditions that:

- $[f, g] \circ \mathsf{inj}_1 = f$ and $[f, g] \circ \mathsf{inj}_2 = g$
- for any $h : A + B \to C$, if $h \circ \mathsf{inj}_1 = f$ and $h \circ \mathsf{inj}_2 = g$, then $h = [f, g]$

A *weak coproduct* of A and B is an object $A + B$ defined alike by omitting the last condition.

A *monad* T on $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ with a family of morphisms $\eta_A : A \to TA$ and $\mu_A : T^2A \to TA$ such that μ is an "associative operation" in the sense that we have $\mu_A \circ \mu_{TA} = \mu_A \circ T\mu_A : T^3A \to TA$ for any object A in $\mathcal{C}$, and η is the "unit" of μ

in the sense that $\mu_A \circ T\eta_A = \mathsf{id}_{TA}$ and $\mu_A \circ \eta_{TA} = \mathsf{id}_{TA}$. Observe that we write T^2A for consecutive applications of T as TTA and similarly for T^3A. Additionally, η and μ must be "natural" in the sense that for any $f : A \to B$ in $\mathcal{C}$, $Tf \circ \eta_A = \eta_B \circ f$ and $Tf \circ \mu_A = \mu_B \circ T^2f$.

A *monad algebra* $\mathfrak{A}_A = (A, \ell_A)$ for a monad T on $\mathcal{C}$, known as a T-algebra, is an object A coupled with an "algebra map" morphism $\ell_A : TA \to A$ in $\mathcal{C}$ such that $\ell_A \circ \eta_A = id_A$ and $\ell_A \circ T\ell_A = \ell_A \circ \mu_A$. A homomorphism h of T-algebras (A, ℓ_A) and (B, ℓ_B) is a morphism $h : A \to B$ that commutes with the algebra maps as $h \circ \ell_A = \ell_B \circ Th$.

Proof of Proposition 40. The morphism μ_A is an algebra map for the object TA when it satisfies the equations $\mu_A \circ \eta_{TA} = \mathsf{id}_{TA}$ and $\mu_A \circ T\mu_A = \mu_A \circ \mu_{TA}$. The former is an instance of the second unit law of the monad T and the latter is instance of the associativity law of T. The functor laws of J similarly follow readily from those of the functor T. ◀

Proof of Lemma 41. It is given that $\mathcal{B}$ is a bicartesian category and T is a monad.

1. We define the cartesian structure of $\mathcal{B}^T$ using the cartesian structure of $\mathcal{B}$. The terminal algebra is given by $(1, \mathsf{term}_1 : T(1) \to 1)$ and the product of two algebras $\mathfrak{A}_A = (A, \ell_A)$ and $\mathfrak{A}_B = (B, \ell_B)$ is given by $\mathfrak{A}_A \times \mathfrak{A}_B = (A \times B, \ell_{A\times B})$ where $\ell_{A\times B} = \langle \ell_A \circ T\mathsf{proj}_1, \ell_B \circ T\mathsf{proj}_2\rangle$. We can show that $\mathfrak{A}_A \times \mathfrak{A}_B$ is an algebra by leveraging the structure of the underlying algebras $\mathfrak{A}_A$ and $\mathfrak{A}_B$, while employing the properties of products in $\mathcal{B}$ and the accompanying monad T where needed. For example, we show the condition $\ell_{A\times B} \circ \eta_{A\times B} = \mathsf{id}_{A\times B}$ on the algebra map $\ell_{A\times B}$ as follows:

$$
\begin{aligned}
&\ell_{A\times B} \circ \eta_{A\times B} \\
&= \langle \ell_A \circ T\mathsf{proj}_1, \ell_B \circ T\mathsf{proj}_2\rangle \circ \eta_{A\times B} \\
&= \langle \ell_A \circ T\mathsf{proj}_1 \circ \eta_{A\times B}, \ell_B \circ T\mathsf{proj}_2 \circ \eta_{A\times B}\rangle \\
&= \langle \ell_A \circ \eta_A \circ \mathsf{proj}_1, \ell_B \circ \eta_B \circ \mathsf{proj}_2\rangle \\
&= \langle \mathsf{id}_A \circ \mathsf{proj}_1, \mathsf{id}_B \circ \mathsf{proj}_2\rangle \\
&= \langle \mathsf{proj}_1, \mathsf{proj}_2\rangle \\
&= \mathsf{id}_{A\times B}
\end{aligned}
$$

   In a similar routine fashion, we can also show $\ell_{A\times B} \circ T\ell_{A\times B} = \ell_{A\times B} \circ \mu_{A\times B}$.

2. $J0 = (T0, \mu_0)$ is a weak initial object since we can define a morphism $\mathsf{init}_{\mathfrak{A}_Z} : J0 \to \mathfrak{A}_Z$ for any algebra $\mathfrak{A}_Z$ as $\mathsf{init}_{\mathfrak{A}_Z} = \ell_Z \circ T\mathsf{init}_Z$. Moreover, $J0$ satisfies $\mathsf{id}_{J0} = \mathsf{init}_{J0}$ as follows:

$$
\begin{aligned}
&\mathsf{id}_{J0} \\
&= \mathsf{id}_{T0} \\
&= \mu_0 \circ T\eta_0 \\
&= \mu_0 \circ T\mathsf{init}_{T0} \\
&= \ell_{T0} \circ T\mathsf{init}_{T0} \\
&= \mathsf{init}_{J0}
\end{aligned}
$$

Similarly, for any homomorphism $h : \mathfrak{A}_A \Rightarrow \mathfrak{A}_Z$ we have $h \circ \mathsf{init}_{\mathfrak{A}_A} = \mathsf{init}_{\mathfrak{A}_Z} : J0 \to \mathfrak{A}_Z$ as:

$$
\begin{aligned}
& h \circ \mathsf{init}_{\mathfrak{A}_A} \\
& = h \circ \ell_A \circ T\mathsf{init}_A \\
& = \ell_Z \circ Th \circ T\mathsf{init}_A \\
& = \ell_Z \circ T(h \circ \mathsf{init}_A) \\
& = \ell_Z \circ T\mathsf{init}_Z \\
& = \mathsf{init}_{\mathfrak{A}_Z}
\end{aligned}
$$

Effectively, we leverage the fact that h is a homomorphism to "commute" arrows until it can be composed as $h \circ \mathsf{init}_A : 0 \to Z$, which must be equivalent to the unique $\mathsf{init}_Z : 0 \to Z$.

3. $J(A + B) = (T(A + B), \mu_{A+B})$ can be shown to be a weak coproduct object since we can define morphisms $\mathsf{inj}_1{}' : A \to T(A + B)$ as $\mathsf{inj}_1{}' = T\mathsf{inj}_1 \circ \eta_A$, $\mathsf{inj}_2{}' : B \to T(A + B)$ as $\mathsf{inj}_2{}' = T\mathsf{inj}_2 \circ \eta_B$, and $[f, g]' : T(A + B) \to C$ as $[f, g]' = \ell_C \circ T[f, g]$. As before with $J0$, we can show $\mathsf{id}_{J(A+B)} = [\mathsf{inj}_1{}', \mathsf{inj}_2{}']'$ by leveraging the uniqueness of $[-, -]$ in $\mathcal{B}$ and monad laws of T, and $h \circ [f, g]' = [h \circ f, h \circ g]'$ by also leveraging ability to commute h.
4. The projection morphisms proj_1 and proj_2 in $\mathcal{B}^T$ can be shown to be homomorphisms by readily leveraging the algebra structure of products in $\mathcal{B}^T$. The morphism $[f, g]' : J(A + B) \to \mathfrak{A}_C$ is a homomorphism for any $f : \mathfrak{A}_A \to \mathfrak{A}_C$, $g : \mathfrak{A}_B \to \mathfrak{A}_C$ since:

$$
\begin{aligned}
& [f, g]' \circ \ell_{T(A+B)} \\
& = [f, g]' \circ \mu_{A+B} \\
& = \ell_C \circ T[f, g] \circ \mu_{A+B} \\
& = \ell_C \circ \mu_C \circ T^2[f, g] \\
& = \ell_C \circ T\ell_C \circ T^2[f, g] \\
& = \ell_C \circ T(\ell_C \circ T[f, g]) \\
& = \ell_C \circ T([f, g]')
\end{aligned}
$$

Similarly the morphism $\mathsf{init}_{\mathfrak{A}_A} : J0 \to \mathfrak{A}_A$ is too a homomorphism:

$$
\begin{aligned}
& \mathsf{init}_{\mathfrak{A}_A} \circ \ell_{T0} \\
& = \mathsf{init}_{\mathfrak{A}_A} \circ \mu_0 \\
& = \ell_A \circ T\mathsf{init}_A \circ \mu_0 \\
& = \ell_A \circ \mu_A \circ T^2\mathsf{init}_A \\
& = \ell_A \circ T\ell_A \circ T^2\mathsf{init}_A \\
& = \ell_A \circ T(\ell_A \circ T\mathsf{init}_A) \\
& = \ell_A \circ T\mathsf{init}_{\mathfrak{A}_A}
\end{aligned}
$$

◀