Cover Semantics for Intuitionistic Modalities

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British Colloquium for Theoretical Computer Science, University of Birmingham, 30 March ‘26

Intuitionistic modal logic

$$\varphi, \psi := \ p, q, \ldots \ \mid \bot \mid \varphi \vee \psi \mid \ \ldots$$
$$\ldots \mid \Box\varphi \mid \Diamond\varphi \mid \blacklozenge\varphi \ldots$$

$\Box\varphi$: "necessarily phi"

$\Diamond\varphi$: "possibly phi"

$\blacklozenge\varphi$: "previously phi"

$\vdots$

Oxidizing OCaml with Modal Memory Management

2024

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LEO WHITE, Jane Street, United Kingdom
STEPHEN DOLAN, Jane Street, United Kingdom
RICHARD A. EISENBERG, Jane Street, USA
SAM LINDLEY,

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Multi-stage Programming with Splice Variables

2025

TSUNG-JU CHIANG, University of Toronto, Canada
NINGNING XIE, University of Toronto, Canada

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Structural Information Flow: A Fresh Look at Types for Non-interference

2025

HEMANT GOUNI, Carnegie Mellon University, USA
FRANK PFENNING, Carnegie Mellon University, USA
JONATHAN ALDRICH, Carnegie Mellon University, USA

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Modal Effect Types

2025

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Fitch-Style Modal Lambda Calculi

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2017–18

Modal Dependent Type Theory and Dependent Right Adjoints

Lars Birkedal[1], Ranald Clouston[2], Bassel Mannaa[3], Rasmus Ejlers

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2018

Implementing a Modal Dependent Type Theory

DANIEL GRATZER, Aarhus University, Denmark
JONATHAN STERLING, Carnegie Mellon University, United States
LARS BIRKEDAL, Aarhus University, Denmark

Modalities are everywh
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type theory MLTT🔒 su
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2019

Multimodal Dependent Type Theory

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2020

Oxidizing OCaml with Modal Memory Management

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Martin-Löf Type
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2024

Fitch-Style Modal Lambda Calculi

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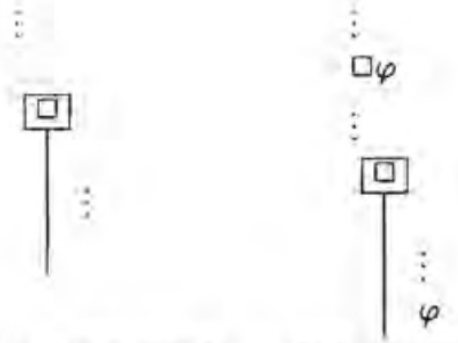
2017–18

Abstract. Fitch-style modal deduction, in which modalities are eliminated by opening a subordinate proof, and introduced by shutting one, were investigated in the 1990s as a basis for lambda calculi. We show that such calculi have good computational properties for a variety of intuitionistic modal logics. Semantics are given in cartesian closed categories equipped with an adjunction of endofunctors, with the necessity modality interpreted by the right adjoint. Where this functor is an idempotent comonad, a coherence result on the semantics allows us to present a calculus for intuitionistic S4 that is simpler than others in the literature. We show the calculi can be extended à la tense logic with the left

Coming to Terms with Modal Logic: On the interpretation of modalities in typed λ-calculus

Tijn Borghuis



A strict subordinate proof *K*-import

K-import: φ may occur in a strict subordinate proof if □φ occurs earlier in the proof to which it is immediately subordinate.

A formula that has been imported into a strict subordinate proof never counts as hypothesis of that proof. Strict subordinate proofs may be written as part of another proof, hence we can have arbitrary nestings of strict and ordinary subordinate proofs.

Formulas can also 'travel' in the opposite direction: conclusions (φ) derived by means of a *categorical* strict subordinate proof may be added to the main proof in a necessitated form ($\Box\varphi$). A subordinate proof is categorical when all its assumptions have been discharged; the conclusion lies directly inside the modal interval, there are no nested subordinate proofs that are still 'open'. This procedure for 'exporting' information from the strict subordinate proof to the main proof is expressed in the following rule:

proof theory has won

what about the model theory?

semantics of IMLs aren’t used in the study of modal type systems why?

an opportunity missed!

Kripke-style relational semantics

$$\mathcal{M} = (W, \sqsubseteq, \ldots, V) \qquad V : \mathrm{Atom} \to \mathcal{P}(W)$$

$$\begin{array}{l} \ldots \\ \mathcal{M}, w \Vdash \bot \qquad \text{iff false} \\ \mathcal{M}, w \Vdash A \vee B \text{ iff } \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B \\ \ldots \end{array}$$

Kripke-style relational semantics

$$\mathcal{M} = (W, \sqsubseteq, R, V) \qquad R \subseteq W \times W$$

$$\begin{aligned}
&\ldots \\
&\mathcal{M}, w \Vdash \Box A \text{ iff } \forall v.\, w\ R\ v \text{ implies } \mathcal{M}, v \Vdash A \\
&\mathcal{M}, w \Vdash \Diamond A \text{ iff } \exists v.\, w\ R\ v \text{ and } \mathcal{M}, v \Vdash A \\
&\ldots
\end{aligned}$$

Kripke-style relational semantics relies on classical reasoning principles

…which doesn’t suit the computational concerns of modal type systems

Goldblatt-style relational "cover" semantics

$$\mathcal{M} = (W, \sqsubseteq, \lhd, \ldots, V) \qquad \lhd \subseteq W \times \mathcal{P}(W)$$

$$\begin{aligned}
&\ldots \\
&\mathcal{M}, w \Vdash \bot \qquad \text{iff } w \lhd \emptyset \\
&\mathcal{M}, w \Vdash A \vee B \text{ iff } \exists \alpha.\, w \lhd \alpha \text{ and } \forall v \in \alpha.\, \mathcal{M}, v \Vdash A \text{ or } \mathcal{M}, v \Vdash B \\
&\ldots
\end{aligned}$$

Goldblatt-style relational "cover" semantics

$$\mathcal{M} = (W, \sqsubseteq, \lhd, \{R_\Box, R_\Diamond, \ldots\}, V)$$

$$\begin{array}{l} \ldots \\ \mathcal{M}, w \Vdash \Box A \text{ iff } \exists v.\, w\ R_\Box\ v \text{ and } \mathcal{M}, v \Vdash A \\ \mathcal{M}, w \Vdash \Diamond A \text{ iff } \exists v.\, w\ R_\Diamond\ v \text{ and } \mathcal{M}, v \Vdash A \\ \ldots \end{array}$$

e.g. Necessitation rule is valid for $\Box$ when $R_\Box$ is serial “there is more to intuitionistic modal logic than the generalisation of properties of boxes and diamonds from Boolean modal logic”

— Goldblatt, *Cover semantics for quantified lax logic*, 2010.

Relational cover semantics bypasses classical reasoning

Relational cover semantics bypasses classical reasoning, but complicates model construction

scope

Goldblatt's interoperability conditions

$$\mathcal{M} = (W, \sqsubseteq, \lhd, R, V)$$

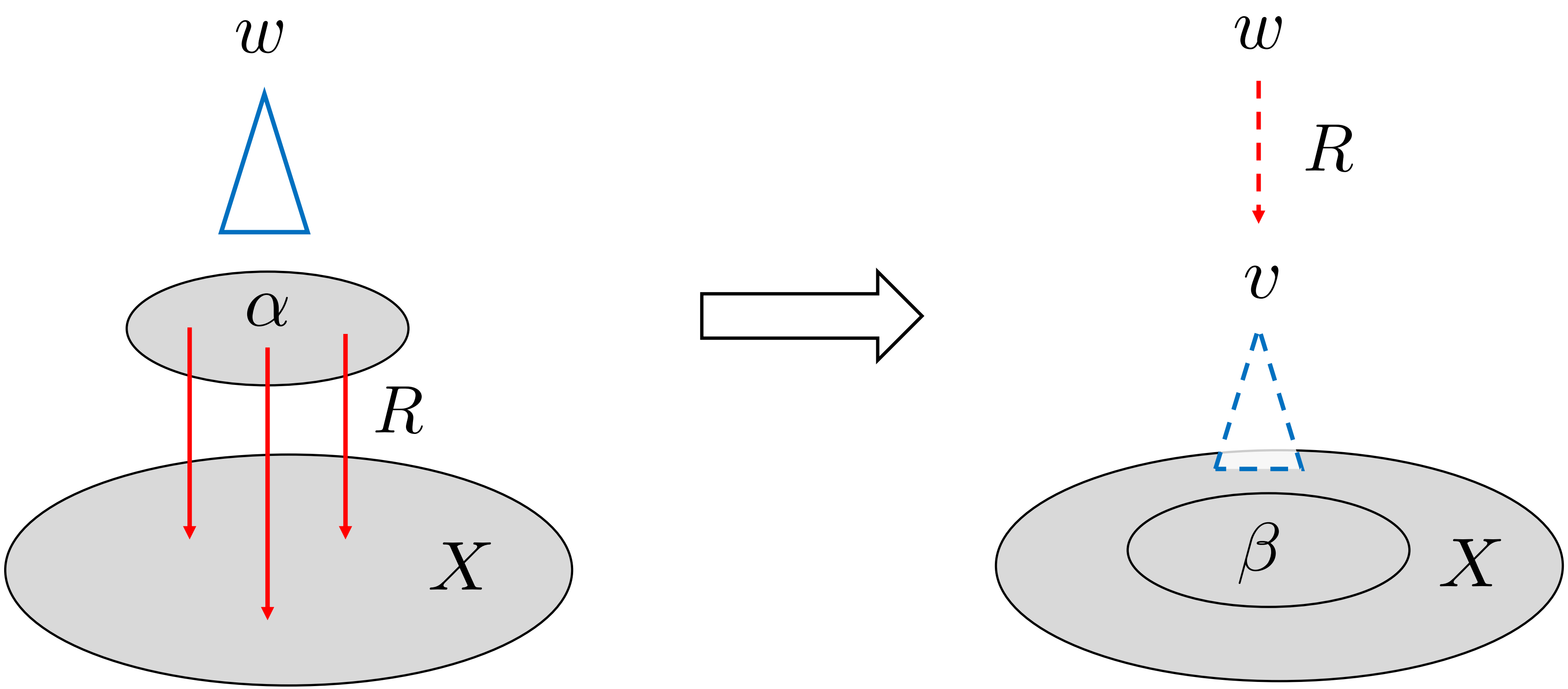
$$\text{If } w' \sqsupseteq w \; R \; v, \text{ there exists a } v' \text{ s.t. } w' \; R \; v' \sqsupseteq v$$

$$\text{If } w \lhd \alpha \subseteq \langle R \rangle X, \text{ there exist } v, \beta \text{ s.t. } w \; R \; v \lhd \beta \subseteq X$$

$$\text{where } \langle R \rangle X = \{ w \in W \mid \exists x \in X. \, w \; R \; x \}$$

→ "Modal Localization"

If $w \lhd \alpha \subseteq \langle R \rangle X$, there exist v, β s.t. $w \; R \; v \lhd \beta \subseteq X$



Cover Semantics for Quantified Lax Logic

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Second, the reader may wonder whether we could have used this construction to prove the completeness of QLL for its cover semantics. A natural binary relation $R_\bigcirc$ on S_p can be defined from $\bigcirc$ by putting $xR_\bigcirc y$ iff $\varphi_x \vdash \bigcirc\varphi_y$, where φ_x, φ_y are chosen generators of x, y as in the proof of Theorem 8.2. Then it can be shown that

$$\bigcirc\varphi \in x \text{ iff for some } y,\ xR_\bigcirc y \text{ and } \varphi \in y,$$

so $|\bigcirc\varphi| = \langle R_\bigcirc \rangle |\varphi|$ and membership of $|\bigcirc\varphi|$ behaves like satisfaction for $\bigcirc$. Moreover, $R_\bigcirc$ is confluent and nuclear. But the stumbling block is the Modal Localisation property, which does not appear to be provable for $R_\bigcirc$. It is for that reason that we resorted to the use of the MacNeille completion to construct models that have this essential property.

Modal Localization doesn’t always hold

…especially for a “Henkin-style” canonical model

Key Idea

$$R \subseteq W \times W$$

$$\Downarrow \text{ replace}$$

$$\blacktriangleleft \subseteq W \times \mathcal{P}(W)$$

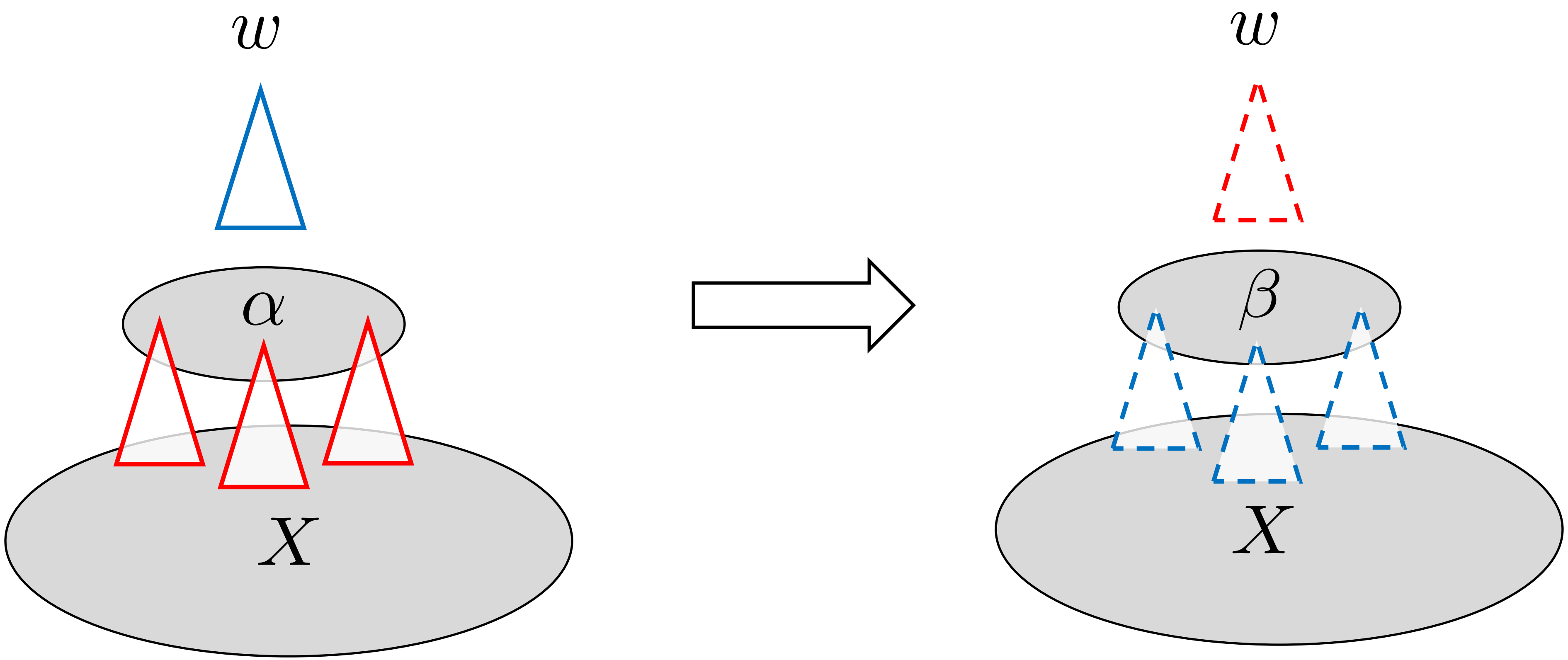
Key Idea

$$\mathcal{M} = (W, \sqsubseteq, \lhd, \not{R}, V)$$

Key Idea

$$\mathcal{M} = (W, \sqsubseteq, \lhd, \blacktriangleleft, V)$$

If $w \lhd \alpha \subseteq \langle \blacktriangleleft \rangle X$, there exists a β s.t. $w \blacktriangleleft \beta \subseteq \langle \lhd \rangle X$

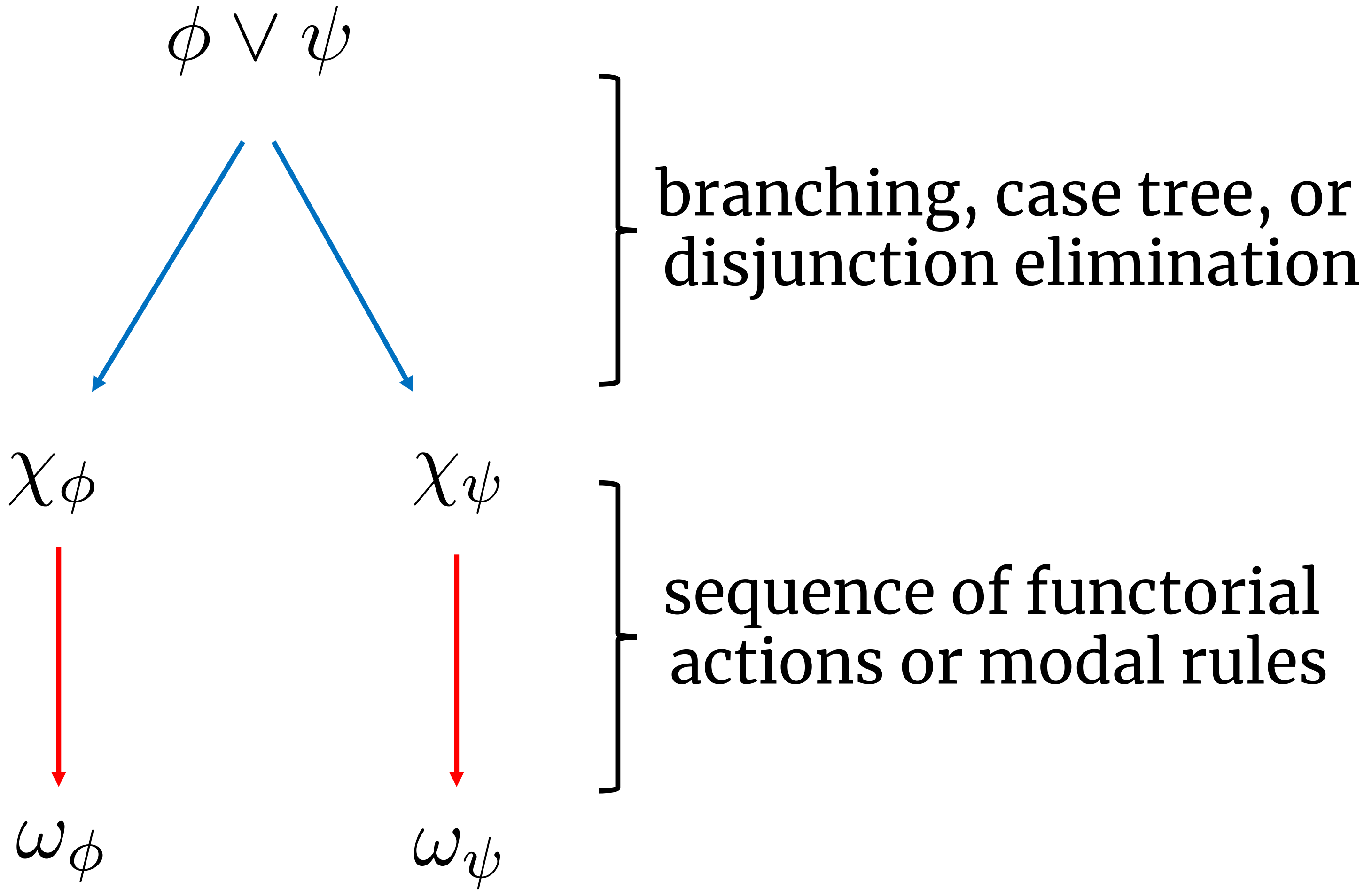


Neighborhood cover semantics bypasses classical reasoning and liberates model construction

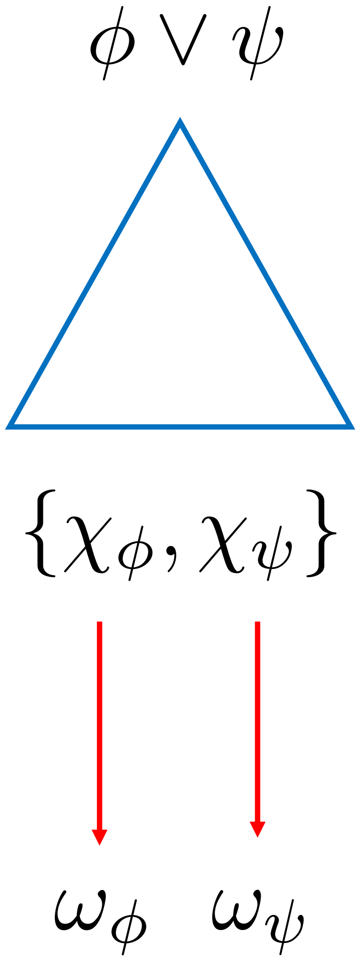
Neighborhood cover semantics is a conservative extension of relational cover semantics p.s. constructivism is the means, not the agenda

Beginning observation: Modal Localization in relational cover semantics blocks construction of Normalization by Evaluation models

e.g. a proof tree



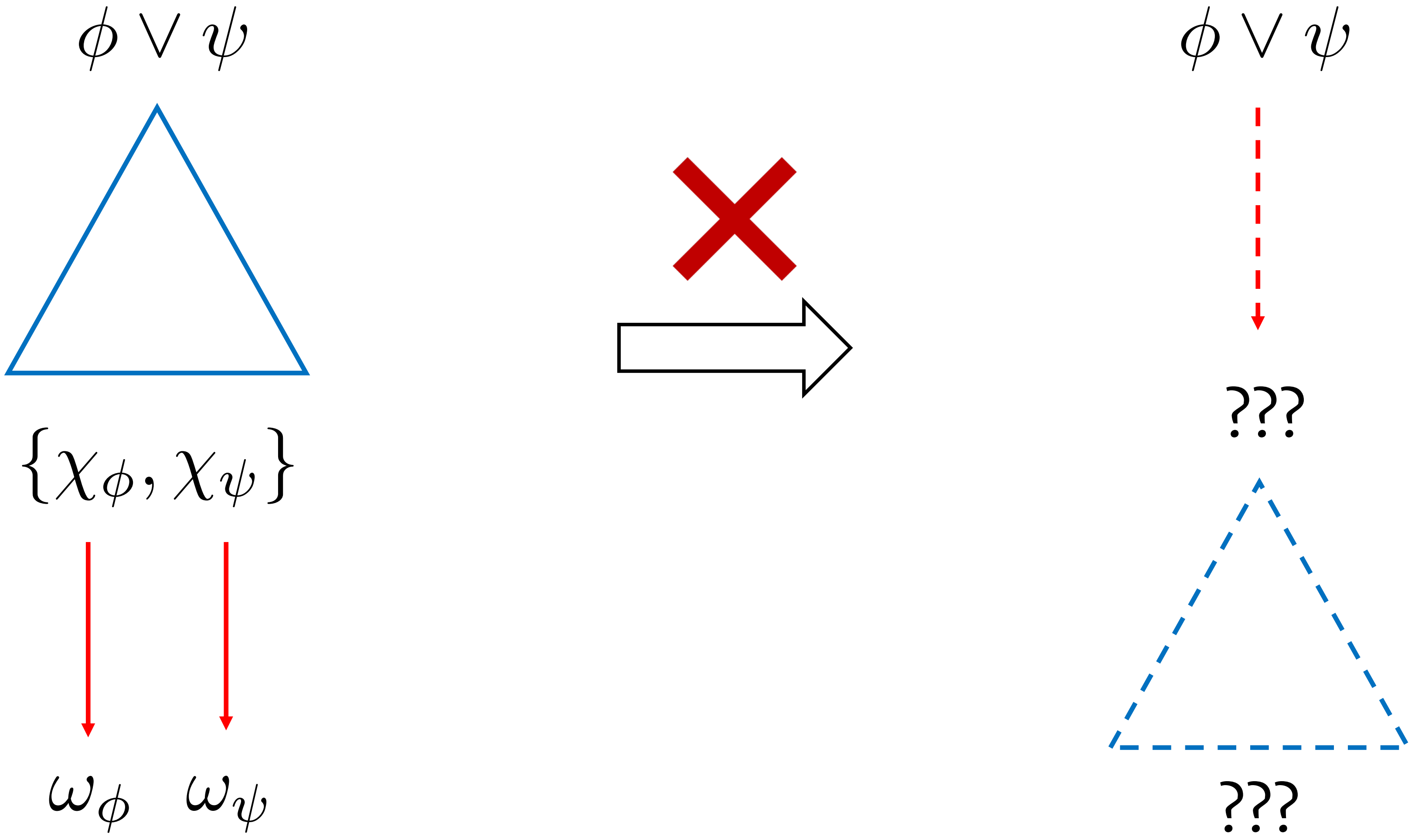
e.g. a proof tree



W = Formulas

$\lhd \sim$ branching

$R \sim$ modal rules



$\phi \vee \psi$
$\{\chi_\phi, \chi_\psi\}$
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$\phi \vee \psi$
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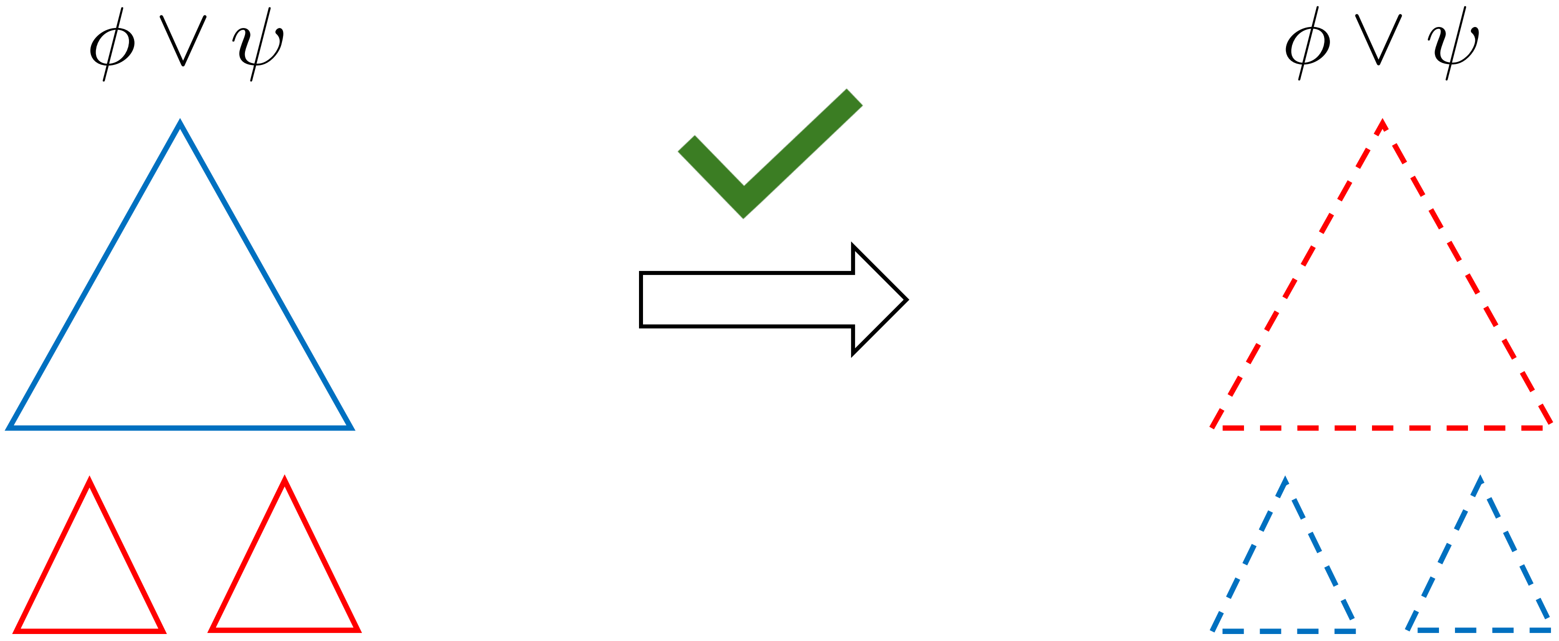
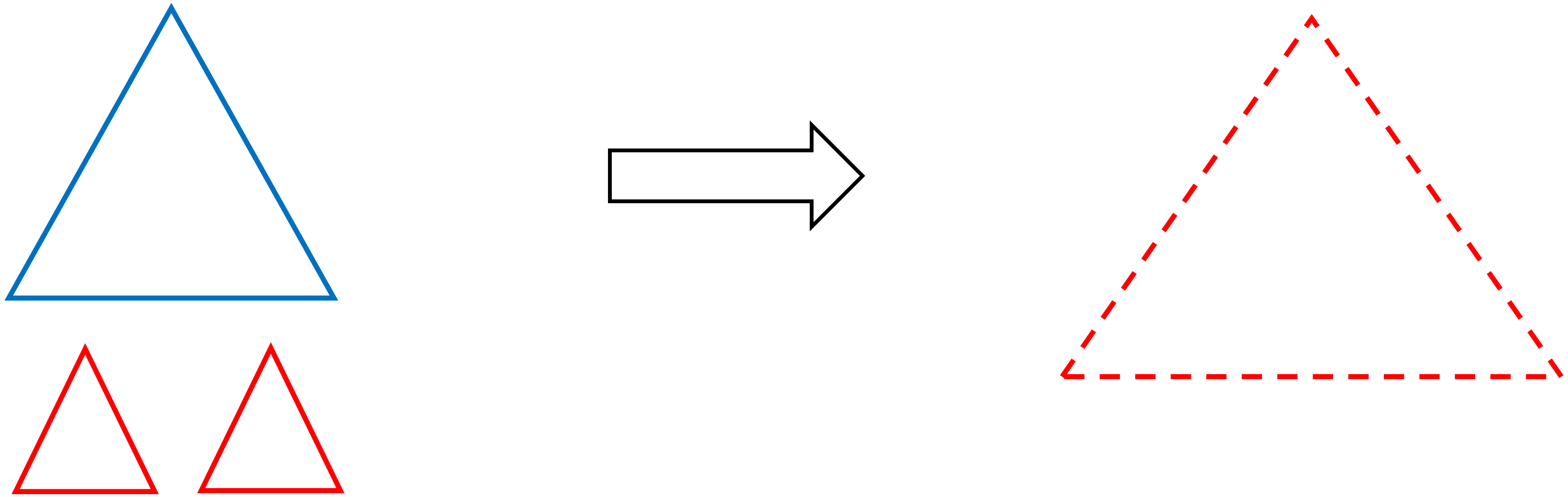
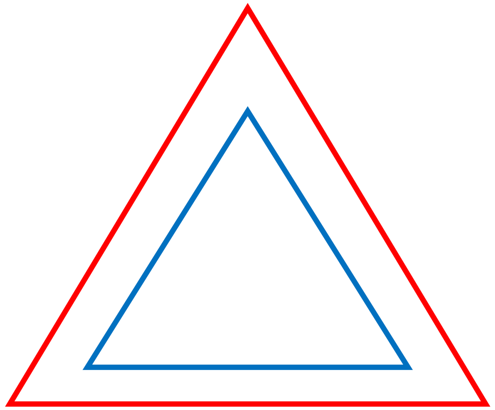

$\phi \vee \psi$
$\phi \vee \psi$

Case study: Absorption in NbE model for CKBox



Case study: Inclusion in NbE model for Lax Logic



Concluding observation: Modal Localization in neighborhood cover semantics allows construction of Normalization by Evaluation models

Theorem 5.1 (Normalization) *Every judgment derivable in the proof system for CM/SL/PLL/CK$_\Box$ has a derivation in normal form. Moreover, every derivation can be normalized to one in normal form.*

Agda

github.com/nachivpn/cover