Cover Semantics for Intuitionistic Modalities

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Intuitionistic modal logic is the study of modalities in intuitionistic logic

$\Box\varphi$: "necessarily phi"

$\Diamond\varphi$: "possibly phi"

$\blacklozenge\varphi$: "previously phi"

$$\vdots$$

This talk is about constructively modelling intuitionistic modalities

Propositional Lax Logic (i.e. logic of Moggi's monadic ML)

Intuitionistic propositional logic +

$$\begin{aligned} \mathrm{S}\colon & A \times \Diamond B \to \Diamond(A \times B) \\ \mathrm{R}\colon & A \to \Diamond A \\ \mathrm{J}\colon & \Diamond\Diamond A \to \Diamond A \end{aligned}$$

Propositional Lax Logic (i.e. logic of Moggi's monadic ML)

Intuitionistic propositional logic +

$$\frac{\Gamma \vdash A}{\Gamma \vdash \lozenge A} \qquad \frac{\Gamma \vdash \lozenge A \qquad \Gamma, A \vdash \lozenge B}{\Gamma \vdash \lozenge B}$$

Kripke's semantics for IPL

$$\mathcal{M} = (W, \sqsubseteq, V) \qquad V : \mathrm{Atom} \to \mathcal{U}(W)$$

$$\begin{aligned}
&\mathcal{M}, w \Vdash p &&\text{iff } w \in V(p)\\
&\mathcal{M}, w \Vdash \top &&\text{iff true}\\
&\mathcal{M}, w \Vdash A \wedge B &&\text{iff } \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B\\
&\mathcal{M}, w \Vdash A \Rightarrow B &&\text{iff } \forall w' \sqsupseteq w.\, \mathcal{M}, w' \Vdash A \text{ implies } \mathcal{M}, w' \Vdash B\\
&\ldots
\end{aligned}$$

Kripke's semantics for IPL

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$$\begin{aligned}
&\ldots \\
&\mathcal{M}, w \Vdash \bot \qquad\quad \text{iff false} \\
&\mathcal{M}, w \Vdash A \vee B \text{ iff } \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B
\end{aligned}$$

Kripke-style relational semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, R, V) \qquad R \subseteq W \times W$$

$$\begin{array}{ll}
\mathcal{M}, w \Vdash p & \text{iff } w \in V(p) \\
\ldots & \\
\mathcal{M}, w \Vdash \bot & \text{iff false} \\
\mathcal{M}, w \Vdash A \vee B & \text{iff } \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B \\
\mathcal{M}, w \Vdash \Diamond A & \text{iff } \exists v.\, w \mathrel{R} v \text{ and } \mathcal{M}, v \Vdash A
\end{array}$$

Soundness and Completeness of Kripke-style semantics

$$\Gamma \vdash A \text{ if and only if } \underbrace{\Gamma \models A}_{\forall \mathcal{M}, w.\, \mathcal{M}, w \Vdash \Gamma \text{ implies } \mathcal{M}, w \Vdash A}$$

Soundness of Kripke-style relational semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, R, V)$$

$$V : \mathrm{Atom} \to \mathcal{U}(W)$$

$$\begin{tikzcd}
w \arrow[r, "R"] \arrow[d, "\sqsubseteq"'] & v \arrow[d, dashed, "\sqsubseteq"] \\
w' \arrow[r, dashed, "R"'] & w
\end{tikzcd}$$

$w \; R \; v$ implies $w \sqsubseteq v$

R reflexive, transitive

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R reflexive, transitive

Every $\mathcal{M}$ determines an algebraic model $\overbrace{\mathcal{M}^+}^{\mathcal{U}(W)}$ of PLL

Completeness for Kripke-style relational semantics is famously not constructive

Constructing a “canonical” model for Kripke-style relational semantics demands classical reasoning

Kripke-style relational semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, R, V) \qquad R \subseteq W \times W$$

$$\begin{aligned}
\mathcal{M}, w \Vdash p \quad & \text{iff } w \in V(p) \\
\ldots & \\
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\mathcal{M}, w \Vdash A \vee B \quad & \text{iff } \mathcal{M}, w \Vdash A \boxed{\text{or}} \mathcal{M}, w \Vdash B \\
\mathcal{M}, w \Vdash \Diamond A \quad & \text{iff } \exists v.\, w \mathrel{R} v \text{ and } \mathcal{M}, v \Vdash A
\end{aligned}$$

…blocking hopes of *categorifying* the semantics to model modal type systems

Beth-Kripke-Joyal "cover" semantics for IPL

$$\mathcal{M} = (W, \sqsubseteq, \lhd, V) \quad \lhd \subseteq W \times \mathcal{P}(W)$$

$\ldots$

$\mathcal{M}, w \Vdash \bot \qquad \text{iff } w \lhd \emptyset$

$\mathcal{M}, w \Vdash A \vee B \text{ iff } \exists \alpha.\, w \lhd \alpha \text{ and } \forall v \in \alpha.\, \mathcal{M}, v \Vdash A \text{ or } \mathcal{M}, v \Vdash B$

Goldblatt's "relational cover" semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, \lhd, R\ , V) \qquad\qquad R \subseteq W \times W$$

$$\begin{array}{ll}
\ldots & \\
\mathcal{M}, w \Vdash \bot & \text{iff } w \lhd \emptyset \\
\mathcal{M}, w \Vdash A \vee B & \text{iff } \exists \alpha.\, w \lhd \alpha \text{ and } \forall v \in \alpha.\, \mathcal{M}, v \Vdash A \text{ or } \mathcal{M}, v \Vdash B \\
\mathcal{M}, w \Vdash \Diamond A & \text{iff } \exists v.\, w\ R\ v \text{ and } \mathcal{M}, v \Vdash A
\end{array}$$

Soundness of relational cover semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, \lhd, R\ , V)$$

$$V : \mathrm{Atom} \to \mathcal{LU}(W)$$

$$\text{i.e. if } \exists \alpha.\, w \lhd \alpha \subseteq V(p), \text{ then } w \in V(p)$$

Every $\mathcal{M}$ determines an algebraic model $\underbrace{\mathcal{M}^+}_{\mathcal{LU}(W)}$ of PLL

Relational cover semantics does not demand classical reasoning

Relational cover semantics does not demand classical reasoning, but severely restricts the class of models

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scope

Relational cover semantics relies upon conditions that prohibit the construction of "Henkin-style" canonical models and Normalization by Evaluation models

Relational cover semantics relies upon conditions that prohibit the construction of

“Henkin-style” canonical models and

Normalization by Evaluation models

desired applications

Goldblatt's interoperability conditions

$$\mathcal{M} = (W, \sqsubseteq, \lhd, R, V)$$

If $w' \sqsupseteq w \; R \; v$, there exists a v' s.t. $w' \; R \; v' \sqsupseteq v$

Goldblatt's interoperability conditions

$$\mathcal{M} = (W, \sqsubseteq, \lhd, R, V)$$

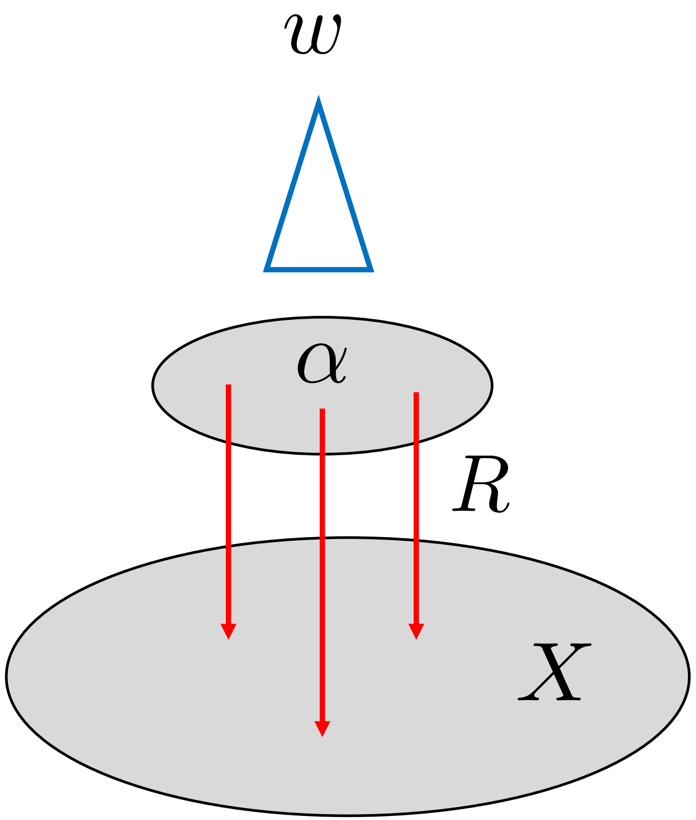
$$\text{If } w' \sqsupseteq w \; R \; v, \text{ there exists a } v' \text{ s.t. } w' \; R \; v' \sqsupseteq v$$

$$\text{If } w \lhd \alpha \subseteq \langle R \rangle X, \text{ there exist } v, \beta \text{ s.t. } w \; R \; v \lhd \beta \subseteq X$$

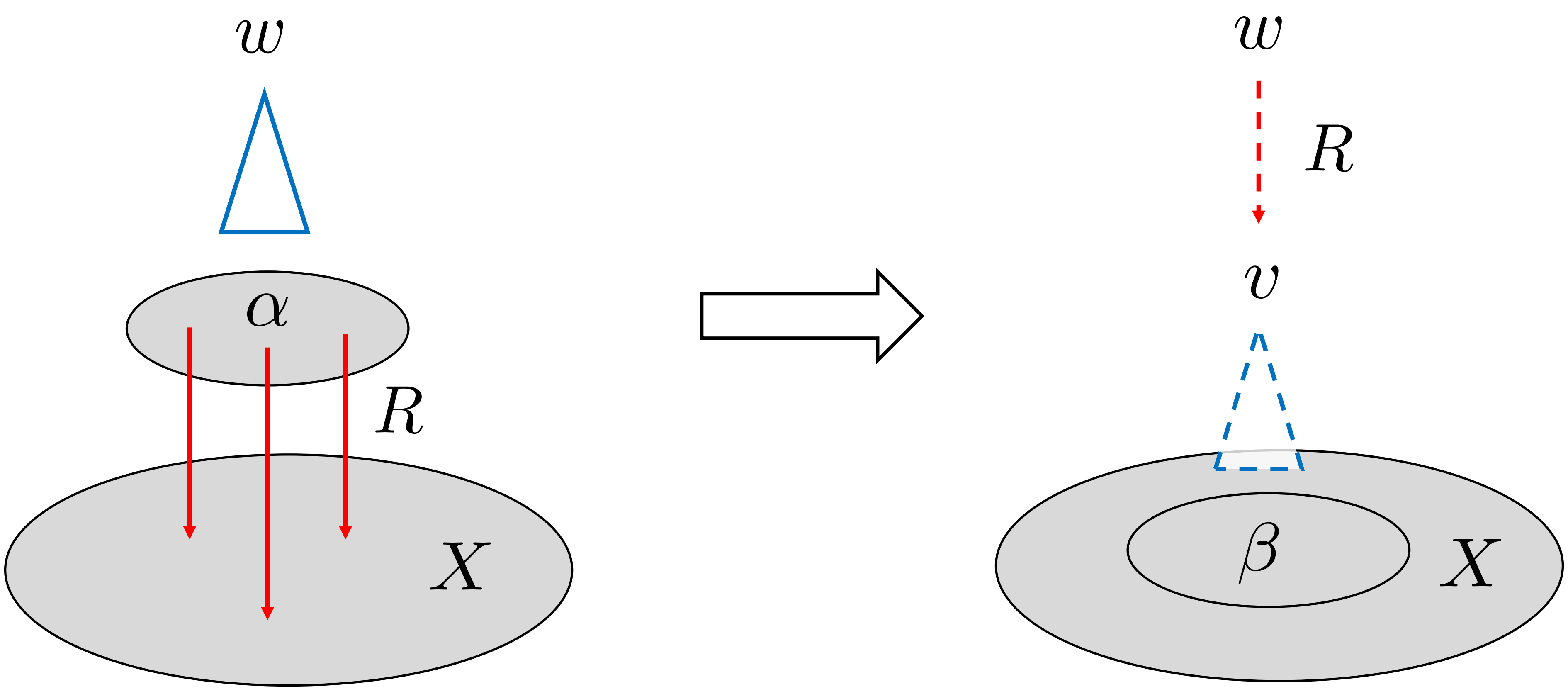
$$\text{where } \langle R \rangle X = \{ w \in W \mid \exists x \in X. \, w \; R \; x \}$$

"Modal Localization"

If $w \lhd \alpha \subseteq \langle R \rangle X$, there exist v, β s.t. $w \; R \; v \lhd \beta \subseteq X$



If $w \lhd \alpha \subseteq \langle R \rangle X$, there exist v, β s.t. $w \; R \; v \lhd \beta \subseteq X$



Cover Semantics for Quantified Lax Logic

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Second, the reader may wonder whether we could have used this construction to prove the completeness of QLL for its cover semantics. A natural binary relation $R_{\bigcirc}$ on S_p can be defined from $\bigcirc$ by putting $xR_{\bigcirc}y$ iff $\varphi_x \vdash \bigcirc\varphi_y$, where φ_x, φ_y are chosen generators of x, y as in the proof of Theorem 8.2. Then it can be shown that

$$\bigcirc\varphi \in x \text{ iff for some } y,\ xR_{\bigcirc}y \text{ and } \varphi \in y,$$

so $|\bigcirc\varphi| = \langle R_{\bigcirc}\rangle|\varphi|$ and membership of $|\bigcirc\varphi|$ behaves like satisfaction for $\bigcirc$. Moreover, $R_{\bigcirc}$ is confluent and nuclear. But the stumbling block is the Modal Localisation property, which does not appear to be provable for $R_{\bigcirc}$. It is for that reason that we resorted to the use of the MacNeille completion to construct models that have this essential property.

Key Idea: Modal covering relation

$$R \subseteq W \times W$$

$$\Big\Downarrow \ \text{replace}$$

$$\blacktriangleleft \subseteq W \times \mathcal{P}(W)$$

Key Idea: Modal covering relation

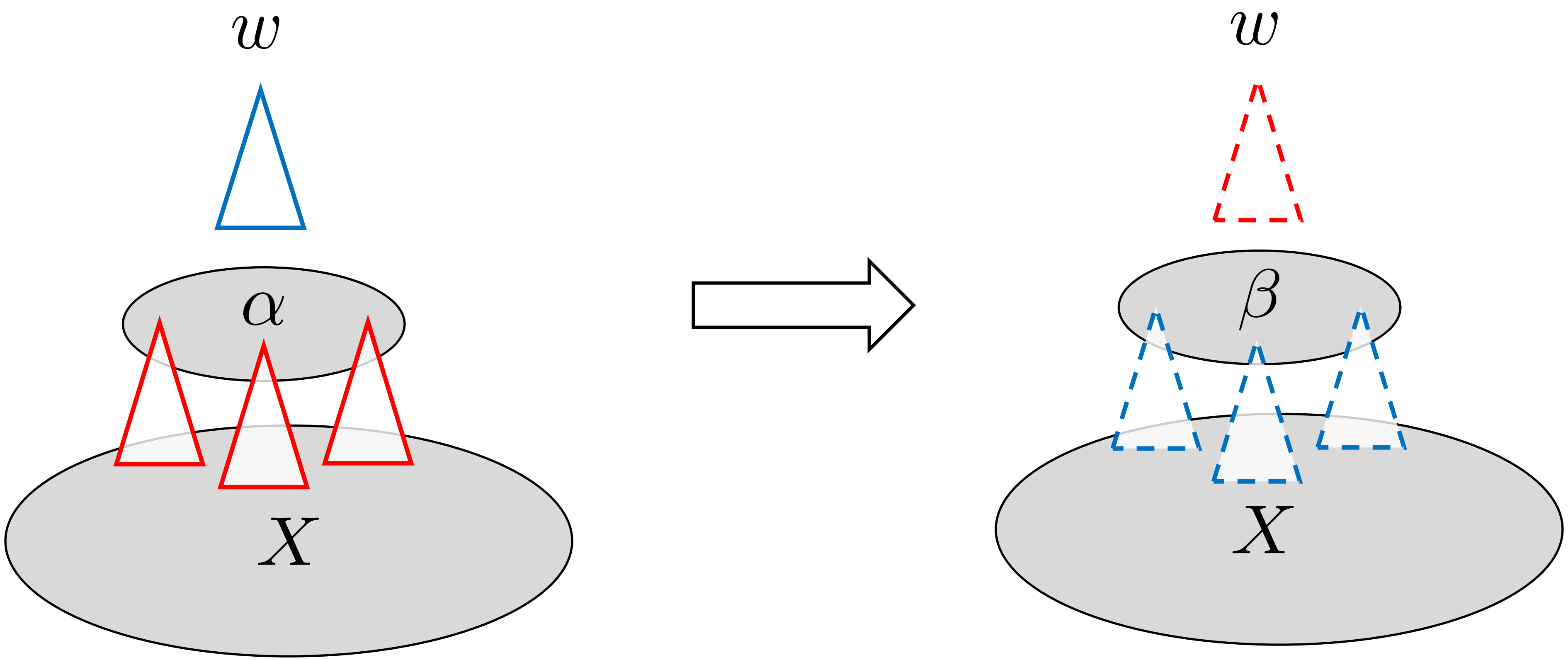
$$\mathcal{M} = (W, \sqsubseteq, \lhd, \blacktriangleleft, V)$$

Modal cover semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, \lhd, \blacktriangleleft, V)$$

$$\begin{array}{ll}
\ldots & \\
\mathcal{M}, w \Vdash \bot & \text{iff } w \lhd \emptyset \\
\mathcal{M}, w \Vdash A \vee B & \text{iff } \exists \alpha.\, w \lhd \alpha \text{ and } \forall v \in \alpha.\, \mathcal{M}, v \Vdash A \text{ or } \mathcal{M}, v \Vdash B \\
\mathcal{M}, w \Vdash \lozenge A & \text{iff } \exists \beta.\, w \blacktriangleleft \beta \text{ and } \forall v \in \beta.\, \mathcal{M}, v \Vdash A
\end{array}$$

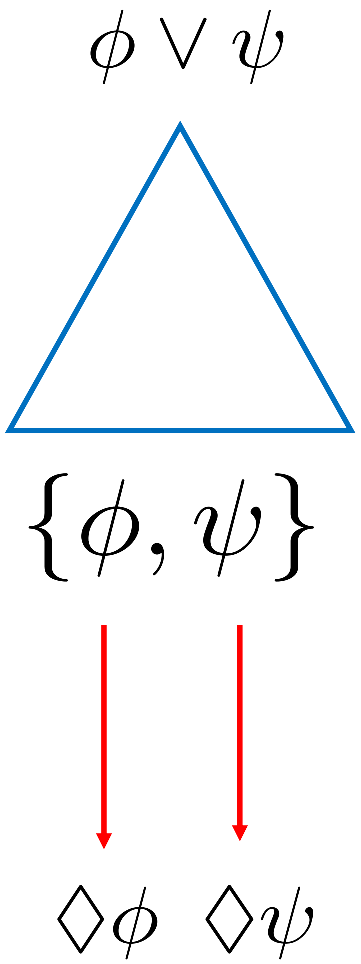
If $w \lhd \alpha \subseteq \langle \blacktriangleleft \rangle X$, there exists a β s.t. $w \blacktriangleleft \beta \subseteq \langle \lhd \rangle X$



Modal cover semantics does not demand classical reasoning and supports a larger class of models; liberating construction

Modal cover semantics is a conservative extension of relational cover semantics

Henkin-style canonical model for PLL



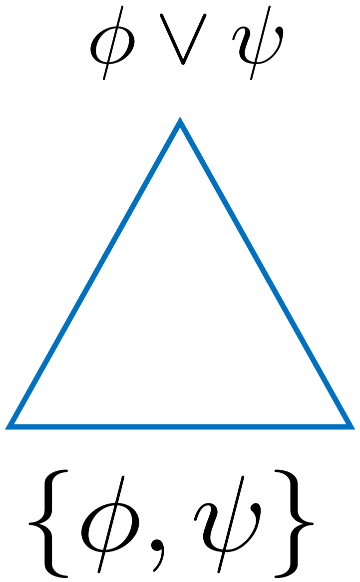
$W = \text{Formulas}$ $\quad$ $A \sqsubseteq B$ iff $B \vdash A$

$A \lhd \alpha$ iff $A \vdash \bigvee \alpha$

$A \, R \, B$ iff $A \vdash \Diamond B$

Henkin-style canonical model for PLL (inductively)

$$W = \text{Formulas} \qquad A \sqsubseteq B \text{ iff } B \vdash A$$



$$\frac{}{A \lhd \{A\}} \qquad \frac{A \vdash \bot}{A \lhd \emptyset}$$

$$\frac{A \vdash A_1 \vee A_2 \quad A_1 \lhd \alpha_1 \quad A_2 \lhd \alpha_2}{A \lhd \alpha_1 \cup \alpha_2}$$

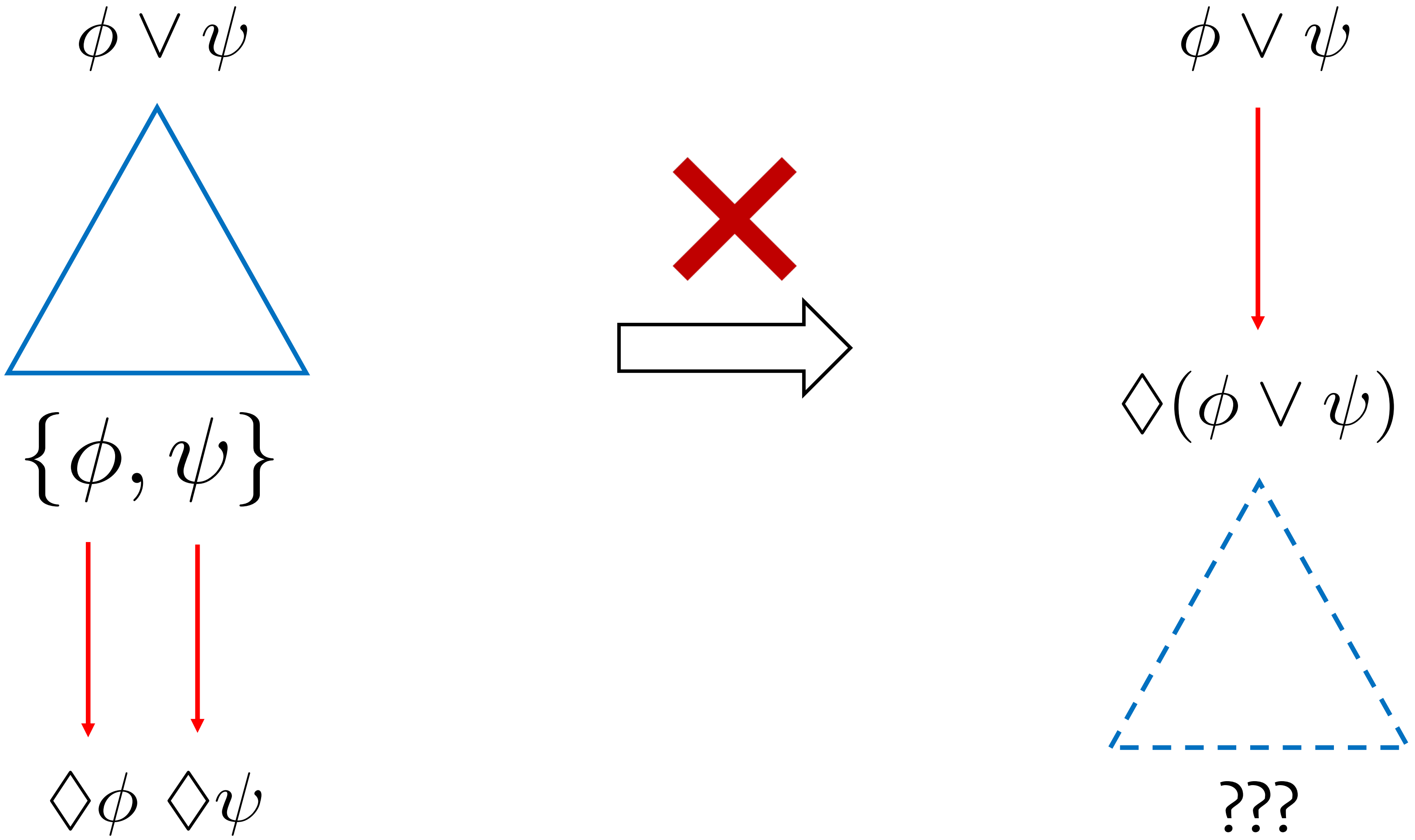
Henkin-style canonical model for PLL (inductively)

$$W = \text{Formulas} \qquad A \sqsubseteq B \text{ iff } B \vdash A$$

$$\{\phi, \psi\} \qquad \frac{}{A\,R\,A} \qquad \frac{A \vdash \Diamond B \quad B\,R\,C}{A\,R\,C}$$

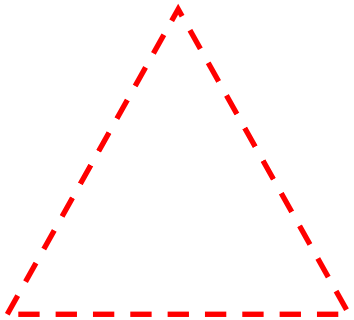
$$\Diamond\phi \quad \Diamond\psi$$

Failure of Modal Localization



Henkin-style canonical model for PLL (inductively)

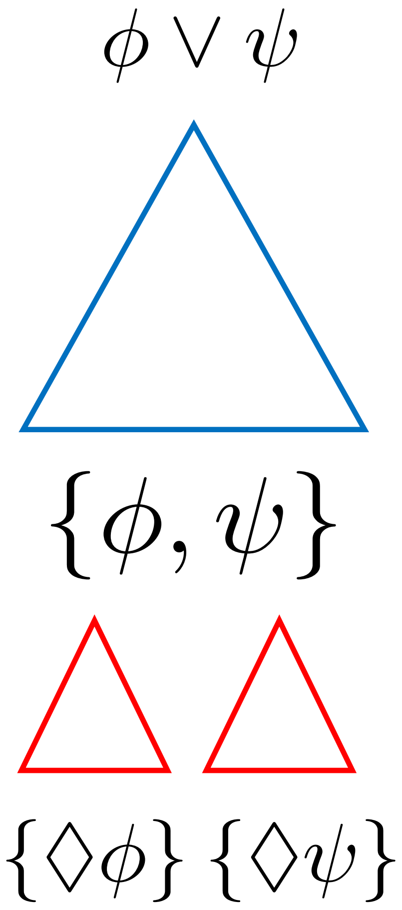
$\phi \vee \psi$



$\{\Diamond\phi, \Diamond\psi\}$

$$\frac{}{A \blacktriangleleft \{A\}} \qquad \frac{A \vdash \Diamond B \quad B \blacktriangleleft \beta}{A \blacktriangleleft \beta}$$

$$\frac{A \vdash \bot}{A \blacktriangleleft \emptyset} \qquad \frac{A \vdash A_1 \vee A_2 \quad A_1 \blacktriangleleft \beta_1 \quad A_2 \blacktriangleleft \beta_2}{A \blacktriangleleft \beta_1 \cup \beta_2}$$


$\phi \vee \psi$
$\{\phi, \psi\}$
$\{\Diamond\phi\}$ $\{\Diamond\psi\}$
$\phi \vee \psi$
$\{\Diamond\phi, \Diamond\psi\}$
$\{\Diamond\phi\}$ $\{\Diamond\psi\}$

Normalization by Evaluation for PLL

$$W = \text{Contexts} \qquad \Gamma \sqsubseteq \Delta \text{ iff } \Gamma \subseteq \Delta$$

$$\frac{}{\Gamma \lhd \{\Gamma\}} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} \bot}{\Gamma \lhd \emptyset}$$

$$\frac{\Gamma \vdash_{\mathrm{NE}} A \vee B \quad \Gamma, A \lhd \alpha_1 \quad \Gamma, B \lhd \alpha_2}{\Gamma \lhd \alpha_1 \cup \alpha_2}$$

Normalization by Evaluation for PLL

$$\frac{}{\Gamma \blacktriangleleft \{\Gamma\}} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} \lozenge A \quad \Gamma, A \blacktriangleleft \alpha}{\Gamma \blacktriangleleft \alpha}$$

$$\frac{\Gamma \vdash_{\mathrm{NE}} \bot}{\Gamma \blacktriangleleft \emptyset} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} A \vee B \quad \Gamma, A \blacktriangleleft \alpha \quad \Gamma, B \blacktriangleleft \beta}{\Gamma \blacktriangleleft \alpha \cup \beta}$$

Main result: “Modal cover semantics” for a family of four modal logics, with a single box or diamond, that supports Henkin-style canonical models and NbE models

Theorem 5.1 (Normalization) *Every judgment derivable in the proof system for CM/SL/PLL/CK*$_\Box$ *has a derivation in normal form. Moreover, every derivation can be normalized to one in normal form.*

Agda

github.com/nachivpn/cover