

Cover Semantics for Intuitionistic Modalities

Nachi Valliappan

University of Edinburgh

ProgLog Seminar, Chalmers University, Gothenburg – Sweden, 27 Oct '26

Intuitionistic modal logic is the study of modalities in intuitionistic logic

$\Box\varphi$: “necessarily phi”

$\Diamond\varphi$: “possibly phi”

$\blacklozenge\varphi$: “previously phi”

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Oxidizing OCaml with Modal Memory Management

ANTON LORENZEN, University of Edinburgh, United Kingdom

LEO WHITE, Jane Street, United Kingdom

STEPHEN DOLAN, Jane Street, United Kingdom

RICHARD A. EISENBERG, Jane Street, USA

SAM LINDLEY

Programmers can
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optimizations can
a design based on

Multi-stage Programming with Splice Variables

TSUNG-JU CHIANG, University of Toronto, Canada

NINGNING XIE, University of Toronto, Canada

2024

2025

Structural Information Flow: A Fresh Look at Types for Non-interference

HEMANT GOUNI, Carnegie Mellon University, USA

FRANK PFENNING, Carnegie Mellon University, USA

JONATHAN ALDRICH, Carnegie Mellon University, USA

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Modal Effect Types

WENHAO TANG, The University of Edinburgh, United Kingdom

LEO WHITE, Jane Street, United Kingdom

STEPHEN DOLAN, Jane Street, United Kingdom

DANIEL HILLERSTRÖM, The University of Edinburgh, United Kingdom

SAM LINDLEY, The University of Edinburgh, United Kingdom

ANTON LORENZEN, The University of Edinburgh, United Kingdom

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2025

2025

Fitch-Style Modal Lambda Calculi

Ranald Clouston^(✉)

2017–18

Department of Computer Science

Modal Dependent Type Theory and Dependent Right Adjoints

Lars Birkedal¹, Ranald Clouston², Bassel Manna³, Rasmus Ejlers

2018

Implementing a Modal Dependent Type Theory

DANIEL GRATZER, Aarhus University, Denmark

JONATHAN STERLING, Carnegie Mellon University, United States

LARS BIRKEDAL, Aarhus University, Denmark

University, Aarhus,

University,

Copenhagen

2019

Modalities are everywhere in type theory MLTT¹ supports a necessity operator. MLTT is a common basis for the proof of the soundness of

Multimodal Dependent Type Theory

Daniel Gratzer
Aarhus University
gratzer@cs.au.dk

G. A. Kavvos
Aarhus University
alex.kavvos@cs.au.dk

2020

Oxidizing OCaml with Modal Memory Management

ANTON LORENZEN, University of Edinburgh, United Kingdom

LEO WHITE, Jane Street, United Kingdom

STEPHEN DOLAN, Jane Street, United Kingdom

Martin-Löf Type Theory
it with new con-

2024

Fitch-Style Modal Lambda Calculi

Ranald Clouston^(✉)

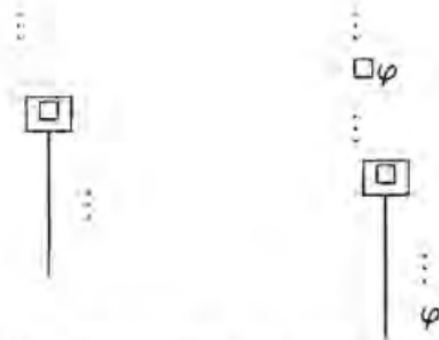
Department of Computer Science, Aarhus University, Aarhus, Denmark
`ranald.clouston@cs.au.dk`

2017–18

Abstract. Fitch-style modal deduction, in which modalities are eliminated by opening a subordinate proof, and introduced by shutting one, were investigated in the 1990s as a basis for lambda calculi. We show that such calculi have good computational properties for a variety of intuitionistic modal logics. Semantics are given in cartesian closed categories equipped with an adjunction of endofunctors, with the necessity modality interpreted by the right adjoint. Where this functor is an idempotent comonad, a coherence result on the semantics allows us to present a calculus for intuitionistic S4 that is simpler than others in the literature. We show the calculi can be extended à la tense logic with the left

Coming to Terms with Modal Logic: On the interpretation of modalities in typed λ -calculus

Tijn Borghuis



A strict subordinate proof K-import

K-import: φ may occur in a strict subordinate proof if $\Box\varphi$ occurs earlier in the proof to which it is immediately subordinate.

A formula that has been imported into a strict subordinate proof never counts as hypothesis of that proof. Strict subordinate proofs may be written as part of another proof, hence we can have arbitrary nestings of strict and ordinary subordinate proofs.

Formulas can also 'travel' in the opposite direction: conclusions (φ) derived by means of a *categorical* strict subordinate proof may be added to the main proof in a necessitated form ($\Box\varphi$). A subordinate proof is categorical when all its assumptions have been discharged; the conclusion lies directly inside the modal interval, there are no nested subordinate proofs that are still 'open'. This procedure for 'exporting' information from the strict subordinate proof to the main proof is expressed in the following rule:

proof theory has won

semantics of IMLs aren't used in
the study of modal type systems

an opportunity missed!

Propositional Lax Logic (i.e. logic of Moggi's monadic ML)

Intuitionistic propositional logic +

$$\text{S: } A \times \Diamond B \rightarrow \Diamond(A \times B)$$

$$\text{R: } A \rightarrow \Diamond A$$

$$\text{J: } \Diamond\Diamond A \rightarrow \Diamond A$$

Propositional Lax Logic (i.e. logic of Moggi's monadic ML)

Intuitionistic propositional logic +

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}$$

$$\frac{\Gamma \vdash \Diamond A \quad \Gamma, A \vdash \Diamond B}{\Gamma \vdash \Diamond B}$$

Kripke's semantics for IPL

$$\mathcal{M} = (W, \sqsubseteq, V) \quad V : \text{Atom} \rightarrow \mathcal{U}(W)$$

$$\mathcal{M}, w \Vdash p \quad \text{iff } w \in V(p)$$

$$\mathcal{M}, w \Vdash \top \quad \text{iff true}$$

$$\mathcal{M}, w \Vdash A \wedge B \quad \text{iff } \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B$$

$$\mathcal{M}, w \Vdash A \Rightarrow B \quad \text{iff } \forall w' \sqsupseteq w. \mathcal{M}, w' \Vdash A \text{ implies } \mathcal{M}, w' \Vdash B$$

...

Kripke's semantics for IPL

$$\mathcal{M} = (W, \sqsubseteq, V) \quad V : \text{Atom} \rightarrow \mathcal{U}(W)$$

...

$$\mathcal{M}, w \Vdash \perp \quad \text{iff false}$$

$$\mathcal{M}, w \Vdash A \vee B \text{ iff } \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B$$

Kripke-style relational semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, R, V) \qquad R \subseteq W \times W$$

$$\mathcal{M}, w \Vdash p \qquad \text{iff } w \in V(p)$$

...

$$\mathcal{M}, w \Vdash \perp \qquad \text{iff false}$$

$$\mathcal{M}, w \Vdash A \vee B \text{ iff } \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B$$

$$\left[\mathcal{M}, w \Vdash \Diamond A \qquad \text{iff } \exists v. w R v \text{ and } \mathcal{M}, v \Vdash A \right]$$

Conditions on Kripke-style relational frames of PLL

$$\mathcal{M} = (W, \sqsubseteq, R, V)$$

$$V : \text{Atom} \rightarrow \mathcal{U}(W)$$

Interoperability condition

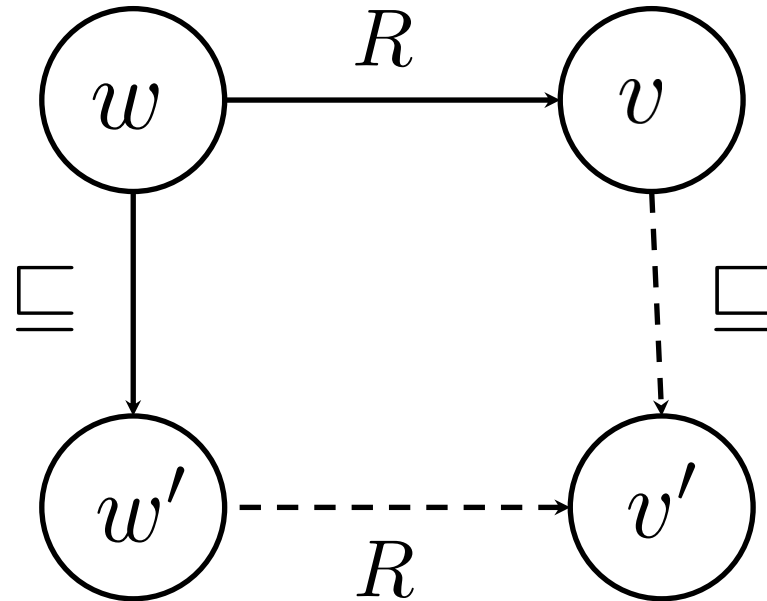
$$\left\{ \begin{array}{l} \text{If } w' \sqsubseteq w \ R \ v, \text{ there exists a } v' \text{ s.t. } w' \ R \ v' \sqsubseteq v \end{array} \right.$$

Additional modal conditions

$$\left\{ \begin{array}{l} w \ R \ v \text{ implies } w \sqsubseteq v \\ R \text{ reflexive, transitive} \end{array} \right.$$

Interoperability condition

If $w' \sqsubseteq w R v$, there exists a v' s.t. $w' R v' \sqsubseteq v$



Soundness and Completeness of Kripke-style semantics

$\Gamma \vdash A$ if and only if $\Gamma \models A$

$\forall \mathcal{M}, w. \mathcal{M}, w \models \Gamma$ implies $\mathcal{M}, w \models A$

Soundness of Kripke-style relational semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, R, V)$$

$$V : \text{Atom} \rightarrow \mathcal{U}(W)$$

Heyting algebra + nucleus

Every $\mathcal{M}$ determines an $\overbrace{\text{algebraic model}}^{\text{Heyting algebra + nucleus}}$ $\mathcal{M}^+$ of PLL

$\underbrace{\text{Carrier of Heyting algebra}}_{\text{Carrier of Heyting algebra given by } \mathcal{U}(W)}$

$$\langle R \rangle X = \{w \in W \mid \exists x \in X. w R x\}$$

Completeness for Kripke-style relational semantics is famously not constructive

Constructing a “canonical” model for
Kripke-style relational semantics
demands classical reasoning

Kripke-style relational semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, R, V) \quad R \subseteq W \times W$$

$$\mathcal{M}, w \Vdash p \quad \text{iff } w \in V(p)$$

...

$$\mathcal{M}, w \Vdash \perp \quad \text{iff } \boxed{\text{false}}$$

$$\mathcal{M}, w \Vdash A \vee B \quad \text{iff } \mathcal{M}, w \Vdash A \boxed{\text{or}} \mathcal{M}, w \Vdash B$$

$$\mathcal{M}, w \Vdash \Diamond A \quad \text{iff } \exists v. w R v \text{ and } \mathcal{M}, v \Vdash A$$

...blocking hopes of *categorifying* the
semantics to model modal type systems

Beth–Kripke–Joyal “cover” semantics for IPL

$$\mathcal{M} = (W, \sqsubseteq, \triangleleft, V) \quad \triangleleft \subseteq W \times \mathcal{P}(W)$$

...

$$\mathcal{M}, w \Vdash \perp \quad \text{iff } w \triangleleft \emptyset$$

$$\mathcal{M}, w \Vdash A \vee B \text{ iff } \exists \alpha. w \triangleleft \alpha \text{ and } \forall v \in \alpha. \mathcal{M}, v \Vdash A \text{ or } \mathcal{M}, v \Vdash B$$

Goldblatt's “relational cover” semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, \triangleleft, R, V)$$

$$R \subseteq W \times W$$

...

$$\mathcal{M}, w \Vdash \perp \quad \text{iff } w \triangleleft \emptyset$$

$$\mathcal{M}, w \Vdash A \vee B \quad \text{iff } \exists \alpha. w \triangleleft \alpha \text{ and } \forall v \in \alpha. \mathcal{M}, v \Vdash A \text{ or } \mathcal{M}, v \Vdash B$$

$$\mathcal{M}, w \Vdash \Diamond A \quad \text{iff } \exists v. w R v \text{ and } \mathcal{M}, v \Vdash A$$

Goldblatt's interoperability conditions

$$\mathcal{M} = (W, \sqsubseteq, \triangleleft, R, V)$$

If $w' \sqsupseteq w \ R \ v$, there exists a v' s.t. $w' \ R \ v' \sqsupseteq v$

Goldblatt's interoperability conditions

$$\mathcal{M} = (W, \sqsubseteq, \triangleleft, R, V)$$

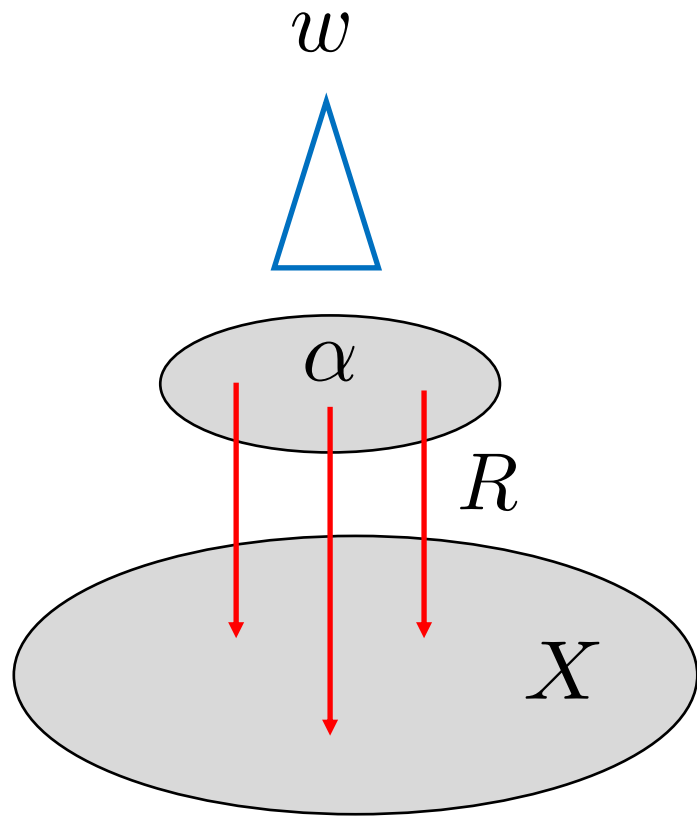
If $w' \sqsupseteq w R v$, there exists a v' s.t. $w' R v' \sqsupseteq v$

If $w \triangleleft \alpha \subseteq \langle R \rangle X$, there exist v, β s.t. $w R v \triangleleft \beta \subseteq X$

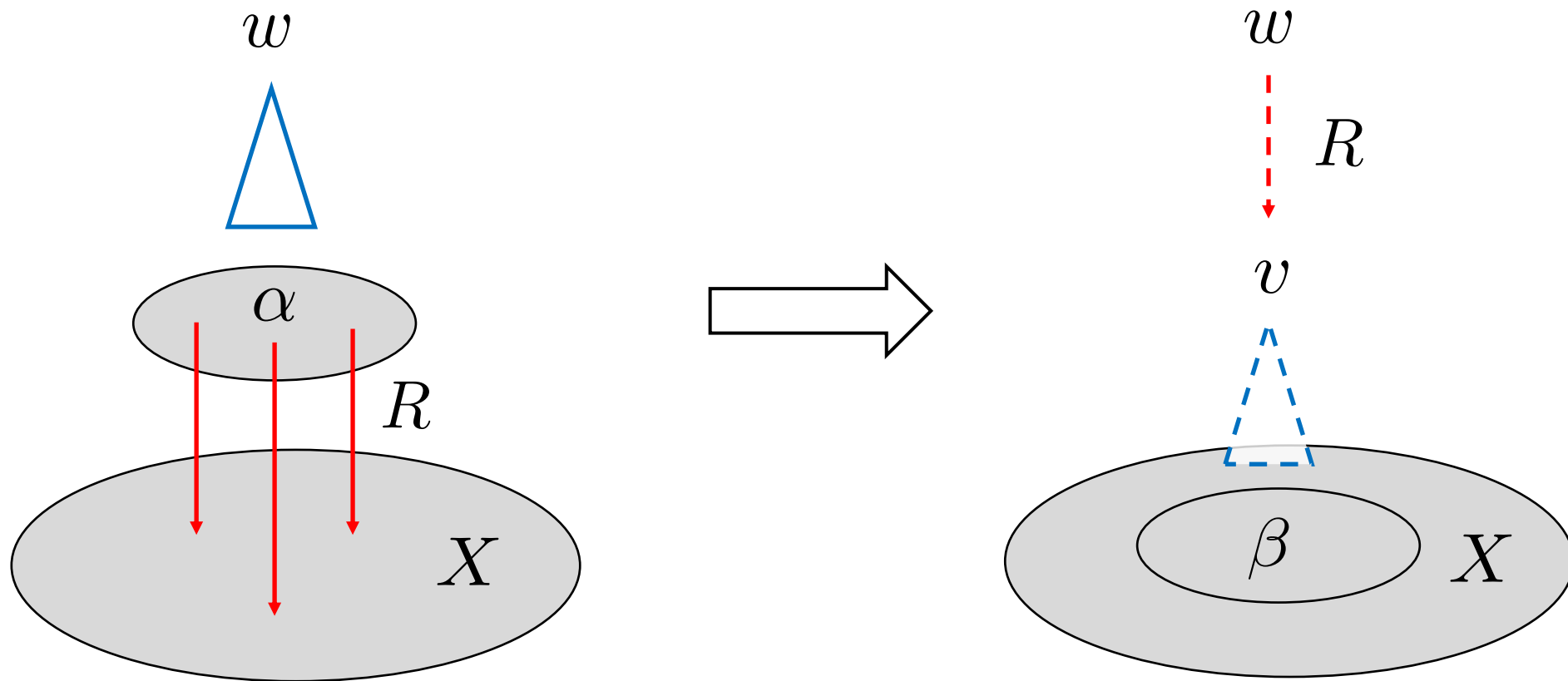
where $\langle R \rangle X = \{w \in W \mid \exists x \in X. w R x\}$

“Modal Localization”

If $w \triangleleft \alpha \subseteq \langle R \rangle X$, there exist v, β s.t. $w R v \triangleleft \beta \subseteq X$



If $w \triangleleft \alpha \subseteq \langle R \rangle X$, there exist v, β s.t. $w R v \triangleleft \beta \subseteq X$



Soundness of relational cover semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, \triangleleft, R, V)$$

$$V : \text{Atom} \rightarrow \mathcal{LU}(W)$$

i.e. if $\exists \alpha. w \triangleleft \alpha \subseteq V(p)$, then $w \in V(p)$

Soundness of relational cover semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, \triangleleft, R, V)$$

$$V : \text{Atom} \rightarrow \mathcal{LU}(W)$$

Every $\mathcal{M}$ determines an algebraic model $\mathcal{M}^+$ of PLL

Carrier of Heyting algebra $\underbrace{\hspace{1.5cm}}$ given by $\mathcal{LU}(W)$

$$\langle R \rangle X = \{w \in W \mid \exists x \in X. w R x\}$$

Relational cover semantics does not
demand classical reasoning

Relational cover semantics does not
demand classical reasoning, but
severely restricts the class of models

Relational cover semantics does not
demand classical reasoning, but
severely restricts the class of models



scope

Modal Localization prohibits construction of
“Henkin-style” canonical models and
Normalization by Evaluation models } desired
applications

Cover Semantics for Quantified Lax Logic

Robert Goldblatt

Centre for Logic, Language and Computation
Victoria University of Wellington, New Zealand
`Rob.Goldblatt@msor.vuw.ac.nz`

Second, the reader may wonder whether we could have used this construction to prove the completeness of QLL for its cover semantics. A natural binary relation $R_\circ$ on S_p can be defined from $\circ$ by putting $xR_\circ y$ iff $\varphi_x \vdash \circ\varphi_y$, where φ_x, φ_y are chosen generators of x, y as in the proof of Theorem 8.2. Then it can be shown that

$$\circ\varphi \in x \text{ iff for some } y, xR_\circ y \text{ and } \varphi \in y,$$

so $|\circ\varphi| = \langle R_\circ \rangle |\varphi|$ and membership of $|\circ\varphi|$ behaves like satisfaction for $\circ$. Moreover, $R_\circ$ is confluent and nuclear. But the stumbling block is the Modal Localisation property, which does not appear to be provable for $R_\circ$. It is for that reason that we resorted to the use of the MacNeille completion to construct models that have this essential property.

Key Idea: Modal covering relation

$$R \subseteq W \times W$$



replace

$$\blacktriangleleft \subseteq W \times \mathcal{P}(W)$$

Key Idea: Modal covering relation

$$\mathcal{M} = (W, \sqsubseteq, \triangleleft, \blacktriangleleft, V)$$

Modal cover semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, \triangleleft, \blacktriangleleft, V)$$

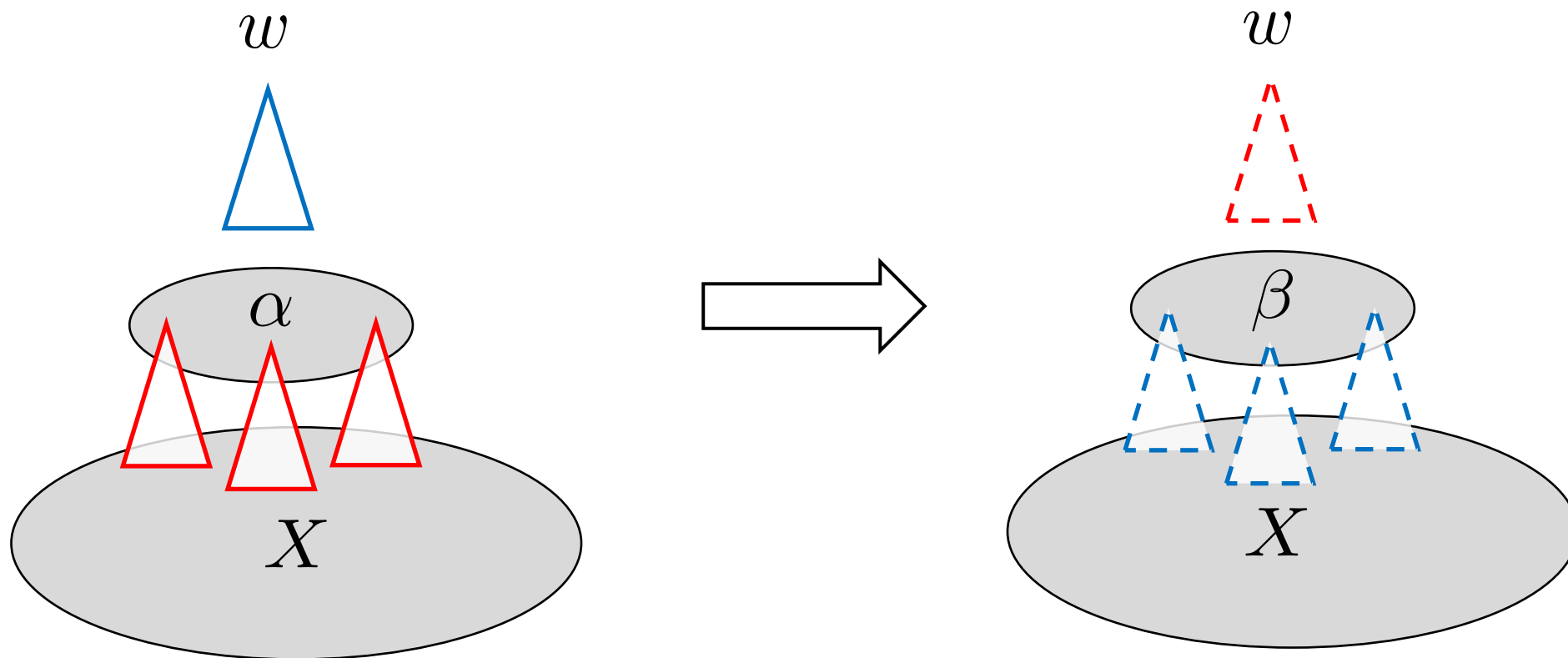
...

$$\mathcal{M}, w \Vdash \perp \quad \text{iff } w \triangleleft \emptyset$$

$$\mathcal{M}, w \Vdash A \vee B \quad \text{iff } \exists \alpha. w \triangleleft \alpha \text{ and } \forall v \in \alpha. \mathcal{M}, v \Vdash A \text{ or } \mathcal{M}, v \Vdash B$$

$$\mathcal{M}, w \Vdash \Diamond A \quad \text{iff } \exists \beta. w \blacktriangleleft \beta \text{ and } \forall v \in \beta. \mathcal{M}, v \Vdash A$$

If $w \triangleleft \alpha \subseteq \langle \blacktriangleleft \rangle X$, there exists a β s.t. $w \blacktriangleleft \beta \subseteq \langle \triangleleft \rangle X$



Modal cover semantics does not demand
classical reasoning and supports a larger
class of models; liberating construction

Modal cover semantics is a conservative
extension of relational cover semantics

Soundness of modal cover semantics for PLL

$$\mathcal{M} = (W, \sqsubseteq, \triangleleft, \blacktriangleleft, V)$$

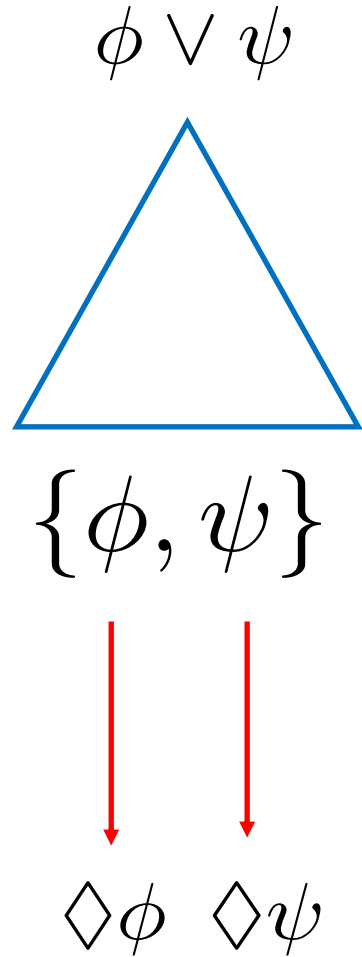
$$V : \text{Atom} \rightarrow \mathcal{LU}(W)$$

Every $\mathcal{M}$ determines an algebraic model $\mathcal{M}^+$ of PLL

Carrier of Heyting algebra $\underbrace{\quad}_{\mathcal{LU}(W)}$ given by $\mathcal{LU}(W)$

$$\langle \blacktriangleleft \rangle X = \{w \in W \mid \exists \alpha. w \blacktriangleleft \alpha \subseteq X\}$$

Henkin-style canonical model for PLL



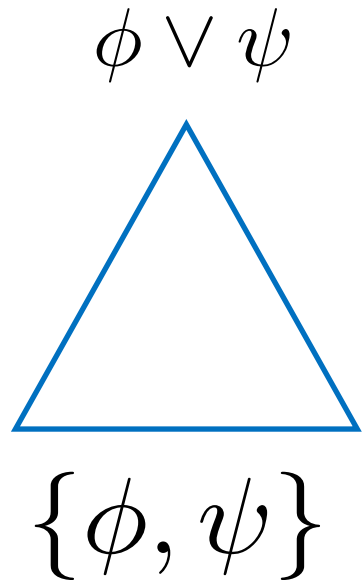
$W = \text{Formulas}$ $A \sqsubseteq B$ iff $B \vdash A$

$A \triangleleft \{B_1, \dots B_n\}$ iff $A \vdash \bigvee \{B_1, \dots B_n\}$
and $B_i \vdash A$

$A R B$ iff $A \vdash \Diamond B$

Henkin-style canonical model for PLL (inductively)

$W = \text{Formulas}$ $A \sqsubseteq B$ iff $B \vdash A$



$$\frac{}{A \triangleleft \{A\}} \qquad \frac{A \vdash \perp}{A \triangleleft \emptyset}$$

$$\frac{A \vdash A_1 \vee A_2 \quad A_1 \triangleleft \alpha_1 \quad A_2 \triangleleft \alpha_2}{A \triangleleft \alpha_1 \cup \alpha_2}$$

Henkin-style canonical model for PLL (inductively)

$$W = \text{Formulas} \quad A \sqsubseteq B \text{ iff } B \vdash A$$

$\{\phi, \psi\}$

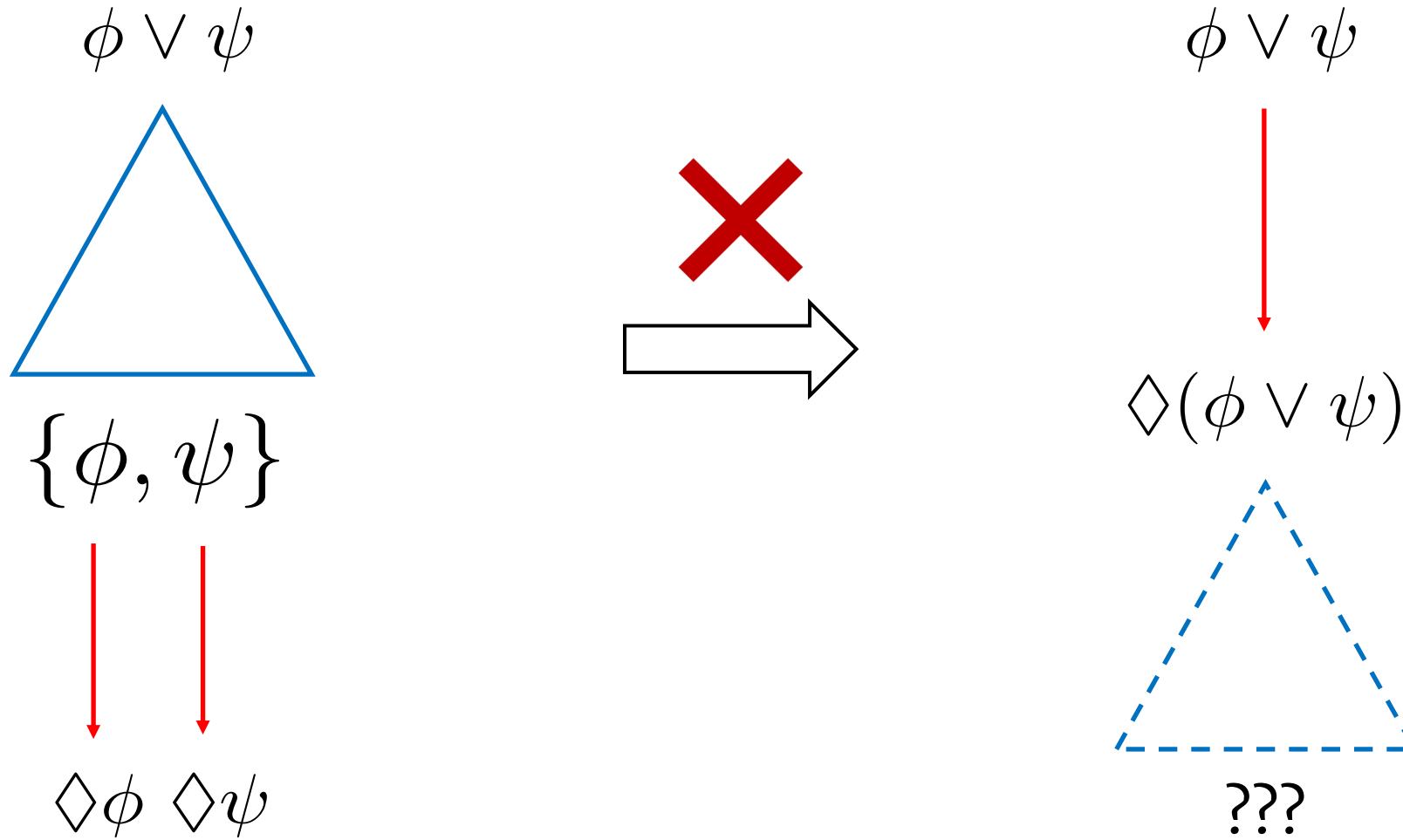


$\Diamond\phi \quad \Diamond\psi$

$$\overline{A R A}$$

$$\frac{A \vdash \Diamond B \quad B R C}{A R C}$$

Failure of Modal Localization

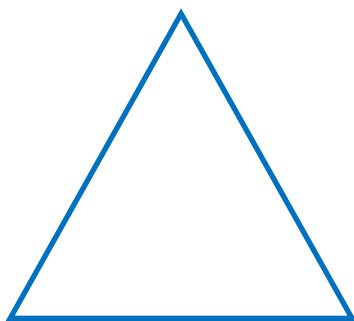


Henkin-style canonical model for PLL (inductively)

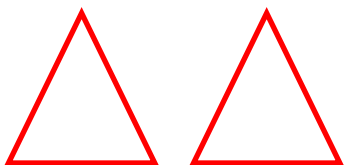
$$\begin{array}{c}
 \phi \vee \psi \\
 \begin{array}{c} \diagup \\ \diagdown \end{array} \\
 \{ \Diamond \phi, \Diamond \psi \}
 \end{array}
 \quad
 \frac{}{A \blacktriangleleft \{A\}}
 \quad
 \frac{A \vdash \Diamond B \quad B \blacktriangleleft \beta}{A \blacktriangleleft \beta}$$

$$\frac{A \vdash \perp}{A \blacktriangleleft \emptyset}
 \quad
 \frac{A \vdash A_1 \vee A_2 \quad A_1 \blacktriangleleft \beta_1 \quad A_2 \blacktriangleleft \beta_2}{A \blacktriangleleft \beta_1 \cup \beta_2}$$

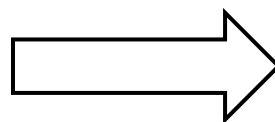
$\phi \vee \psi$



$\{\phi, \psi\}$



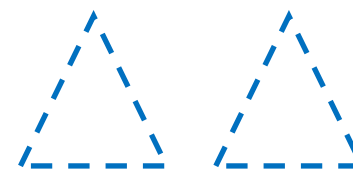
$\{\Diamond\phi\} \{\Diamond\psi\}$



$\phi \vee \psi$



$\{\Diamond\phi, \Diamond\psi\}$



$\{\Diamond\phi\} \{\Diamond\psi\}$

Normalization by Evaluation for PLL

$W = \text{Contexts}$ $\Gamma \sqsubseteq \Delta \text{ iff } \Gamma \subseteq \Delta$

$$\frac{}{\Gamma \triangleleft \{\Gamma\}} \qquad \frac{\Gamma \vdash_{\text{NE}} \perp}{\Gamma \triangleleft \emptyset}$$

$$\frac{\Gamma \vdash_{\text{NE}} A \vee B \quad \Gamma, A \triangleleft \alpha_1 \quad \Gamma, B \triangleleft \alpha_2}{\Gamma \triangleleft \alpha_1 \cup \alpha_2}$$

Normalization by Evaluation for PLL

$$\frac{}{\Gamma \blacktriangleleft \{\Gamma\}} \qquad \frac{\Gamma \vdash_{\text{NE}} \Diamond A \quad \Gamma, A \blacktriangleleft \alpha}{\Gamma \blacktriangleleft \alpha}$$

$$\frac{\Gamma \vdash_{\text{NE}} \perp}{\Gamma \blacktriangleleft \emptyset} \qquad \frac{\Gamma \vdash_{\text{NE}} A \vee B \quad \Gamma, A \blacktriangleleft \alpha \quad \Gamma, B \blacktriangleleft \beta}{\Gamma \blacktriangleleft \alpha \cup \beta}$$

Main result: “Modal cover semantics” for a family of five modal logics, with a single box or diamond, that supports Henkin-style canonical models and NbE models

Theorem 5.1 (Normalization) *Every judgment derivable in the proof systems for CM, SL, PLL, $CK_{\square}$, and $CS4_{\square}$, has a derivation in normal form that can be constructed when given the original derivation.*



github.com/nachivpn/cover

Cover Semantics for Intuitionistic Modalities

Nachiappan Valliappan¹

*School of Informatics
University of Edinburgh
Edinburgh, Scotland*

Abstract

Intuitionistic modal logic (IML) has inspired several developments in programming languages including modal type systems for staging, computational effects and language-based security. IMLs are typically studied using Kripke-style relational semantics, which simplifies proofs of meta-theoretic properties, such as completeness and consistency, by making it easy to construct models. Kripke-style relational semantics, however, relies upon classical reasoning principles, which makes it unappealing from a computational perspective and unsuitable for formalization in a constructive type theory. Goldblatt provides an alternative semantics for IMLs by extending Beth-Kripke-Joyal-style “cover” semantics for intuitionistic propositional logic with relations to support modalities. Goldblatt’s “relational cover” semantics overcomes classical reasoning but introduces a new limitation: it relies upon a “modal localization” condition that restricts the class of models and complicates model construction. Goldblatt bypasses this restriction by using intricate order-theoretic completion arguments to prove completeness. In this article, we present a conservative extension of relational cover semantics that alleviates this restriction and is amenable to simpler and standard model construction techniques. We formalize our semantics in Agda and prove completeness constructively in the style of Normalization by Evaluation for a variety of IMLs featuring independent box and diamond modalities.

Keywords: constructive completeness, intuitionistic modal logic, normalization by evaluation

Cover semantics beyond distributivity

Jim de Groot ✉ 

Mathematical Institute, University of Bern, Bern, Switzerland

Nachiappan Valliappan ✉ 

School of Informatics, University of Edinburgh, Edinburgh, Scotland

Abstract

Cover semantics is an approach to modelling non-classical logics, pioneered by Goldblatt, that can be understood as a generalisation of Beth-Kripke-Joyal semantics for intuitionistic propositional logic. The versatility of cover semantics has been demonstrated in recent results that show how cover semantics can be used to constructively prove meta-logical results such as completeness, consistency and normalisation for a variety of logics with an intuitionistic base. In this work, we take a step backwards and show that this versatility runs deeper than previously shown. We show that cover semantics can as well be used for logics that forego the intuitionistically characteristic distribution of conjunction over disjunction. We begin with cover semantics for a non-distributive positive logic, i.e. without implication or negation, known as lattice logic and show how it can be extended to model complex systems with a lattice base including a positive modal logic, first-order lattice logic, and a typed program calculus with commuting conversions that corresponds to lattice logic.

(whiteboard discussion on future work)