Cover Semantics for Fitch-Style Modal Natural Deduction

Ian Shillito[1,2] and Nachiappan Valliappan[3]

[1] University of Birmingham, Birmingham, United Kingdom
[2] University of Melbourne, Melbourne, Australia
i.b.p.shillito@bham.ac.uk
[3] University of Edinburgh, Edinburgh, United Kingdom
nachivpn@gmail.com

Intuitionistic modal logics typically extend intuitionistic logic with one or both of the (dual) modalities $\Box$ and $\Diamond$. In the last decades, these logics have found myriad applications in notably AI [6], database theory [14], concurrency theory [19] and the design of type systems for programming languages [16, 12, 4]. But their study is notoriously hard, as it combines the subtleties of intuitionistic logic IPL with the complexity of modalities. In this work, we are interested in the $\Box$-only logic $\mathsf{CK}_\Box$, which extends IPL with the following axiom K and *necessitation* rule:

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \qquad\qquad \frac{\emptyset \vdash \varphi}{\Gamma \vdash \Box\varphi}$$

Intuitionistic propositional logic famously corresponds to the simply-typed lambda calculus, with the formulas and proofs of the logic respectively matched to the types and terms of the lambda calculus: this is the celebrated *Curry-Howard correspondence* [7, 18]. Through the extension of this correspondence, several intuitionistic modal logics have been found to correspond to *modal lambda calculi* [1, 2]. A notable Curry-Howard correspondence exists between the logic $\mathsf{CK}_\Box$ and a modal lambda calculus due to Borghuis [3]. This calculus, which can be viewed as a natural deduction system $\mathsf{FCK}_\Box$, was later given a categorical foundation and a refined presentation by Clouston [4].

The Curry-Howard correspondence between the logic $\mathsf{CK}_\Box$ and Borghuis' calculus allows for sharing of tools: machinery to study the logic can be leveraged to investigate the lambda calculus, and vice-versa. One such tool is Kripke-style birelational semantics for $\mathsf{CK}_\Box$. By extending the usual concept of a preordered frame $(X, \sqsubseteq)$ in Kripke's semantics for IPL with a binary relation R, Kripke-style birelational semantics allows us to interpret the $\Box$ modality of $\mathsf{CK}_\Box$, for a model $\mathcal{M}$ induced by the frame, in the following way:

$$\mathcal{M}, w \Vdash \Box\varphi \qquad \text{iff} \qquad \forall w', v.\, w \sqsubseteq w' \text{ and } w' \; R \; v \text{ implies } \mathcal{M}, v \Vdash \varphi$$

It can be shown that $\mathsf{CK}_\Box$ tightly corresponds to the local consequence relation $\models$ over these frames [15, 5], as witnessed by the following soundness-and-completeness result: $\Gamma \vdash \varphi \Leftrightarrow \Gamma \models \varphi$.

While one can use this semantics to study $\mathsf{FCK}_\Box$, we identify two deficiencies in this approach. First, the establishment of completeness proofs with respect to Kripke-style semantics are well-known to readily require non-constructive principles [13, 17]. While non-constructivity can be seen as problematic in itself, it also prevents the algorithmic use of the semantics critical for *normalisation by evaluation* (NbE)—a goal we wish to reach for $\mathsf{FCK}_\Box$. In our specific case, the decidability of $\mathsf{CK}_\Box$ [11] and our focus on finitary contexts should allow for a constructive proof of completeness similar to Hagemeier and Kirst's [10]. Still, this trick becomes out of reach once the logic turns undecidable, e.g. in the presence of quantifiers, making this semantics a poor foundation for the study of extensions of $\mathsf{CK}_\Box$. Second, the above Kripke-style semantics does not provide an insightful account of the rules in $\mathsf{FCK}_\Box$: it does not explain the context modifications present in the rules for $\vee$ and $\bot$, as we elaborate below.

Our work aims to develop a Beth-Kripke-Joyal-style *cover* semantics [8, 9] for $\mathsf{FCK}_\Box$ addressing both deficiencies, by allowing for constructive completeness (and NbE) while giving a semantic account of the unusual rules in $\mathsf{FCK}_\Box$.

The Proof System $\mathsf{FCK}_\Box$ The formal language of the logic $\mathsf{CK}_\Box$ consists of formulas defined inductively by propositional atoms (p, q, r, etc.), constants $\top$ and $\bot$, binary connectives $\wedge, \vee, \Rightarrow$ and a unary connective $\Box$.

$$\text{Prop} \quad A, B := p, q, r, \ldots \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \Rightarrow B \mid \Box A \qquad \text{Ctx} \quad \Gamma, \Delta := \cdot \mid \Gamma, A \mid \Gamma, 🔒$$

The proof system $\mathsf{FCK}_\Box$ [4] is a sequent-style natural deduction calculus for $\mathsf{CK}_\Box$, and is defined inductively as usual using inference rules for judgements $\Gamma \vdash A$ that relate a *context* Γ to a formula A. A context Γ in a judgement of $\mathsf{FCK}_\Box$ is a finite sequence $A_1, ..., A_i, 🔒, A_j, ..., A_k, 🔒, A_m, ..., A_n$ consisting of sequences of formulas delimited by a constant symbol 🔒, with $\cdot$ denoting the empty sequence. A selection of characteristic inference rules for $\mathsf{FCK}_\Box$ are given below. The omitted introduction and elimination rules for the remaining connectives are defined as usual for IPL and can be found in the Appendix. Observe, in particular, how the rules ($\bot$-E) and ($\vee$-E) depart from the usual formulation in IPL and allow the context in the conclusion to be extended with an arbitrary context Γ'.

$$\frac{A \in \Gamma \quad \text{🔒} \notin \Gamma'}{\Gamma, \Gamma' \vdash A}\ (\text{Hyp}) \qquad \frac{\Gamma, \text{🔒} \vdash A}{\Gamma \vdash \Box A}\ (\Box\text{-I}) \qquad \frac{\Gamma \vdash \Box A \quad \text{🔒} \notin \Gamma'}{\Gamma, \text{🔒}, \Gamma' \vdash B}\ (\Box\text{-E}) \qquad \frac{\Gamma \vdash \bot}{\Gamma, \Gamma' \vdash A}\ (\bot\text{-E})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee\text{-I}_1) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee\text{-I}_2) \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A, \Gamma' \vdash C \quad \Gamma, B, \Gamma' \vdash C}{\Gamma, \Gamma' \vdash C}\ (\vee\text{-E})$$

Cover Semantics for $\mathsf{FCK}_\Box$ A cover system $C = (W, \sqsubseteq, \lhd, R)$ is a tuple consisting of a set W of worlds, a preorder $\sqsubseteq$ on W, a *covering* relation $\lhd \subseteq W \times \mathcal{P}(W)$, and a *modal accessibility* relation $R \subseteq W \times W$, all subject to certain compatibility conditions (see Appendix for full list). Given an operator $[R] : \mathcal{P}(W) \to \mathcal{P}(W)$ defined as $[R]X = \{w \mid \forall w', v.\, w \sqsubseteq w' \text{ and } w' \mathrel{R} v \text{ implies } v \in X\}$, the conditions used to model the modal fragment are:

- *Modal Localization*: If $w \lhd \alpha \subseteq [R]X$, $w \sqsubseteq w'$ and $w' \mathrel{R} v$, then $\exists \beta.\, v \lhd \beta \subseteq X$.
- *Lock Refinability*: If $w \mathrel{R} v$ and $v \sqsubseteq v'$, then $\exists w'.\, w \sqsubseteq w' \mathrel{R} v'$.

We say that a subset $X \subseteq W$ is an *up-set* when it satisfies the condition that if $w \sqsubseteq w'$ and $w \in X$ then $w' \in X$, *localized* when $\exists \alpha.\, w \lhd \alpha \subseteq X$ implies $w \in X$, and a *localized up-set* when it satisfies both conditions. The above conditions respectively ensure that *truth sets* of formulas are localized and truth sets of contexts are up-sets, as we will see in Lemma 1, and allow us to prove soundness of cover semantics for judgments in $\mathsf{FCK}_\Box$.

A *cover model* $\mathcal{M} = (C, V)$ for $\mathsf{FCK}_\Box$ couples a cover system C with a valuation function V mapping propositional atoms to localized up-sets of W. This means, for an atom p, $V(p)$ must be an up-set of the preorder $(W, \sqsubseteq)$, and that if p is "locally true" at w, i.e. at all worlds in some cover α of w, then it must be true at w. Given a cover model $\mathcal{M}$ and a world w, the truth of a formula A is defined using the *satisfaction* relation $\mathcal{M}, w \Vdash A$, and extended to contexts via the relation $\mathcal{M}, w \Vdash \Gamma$ as shown below. Note the backward-looking interpretation of 🔒.

$$\begin{array}{lll} \mathcal{M}, w \Vdash p & \text{iff} & w \in V(p) \\ \mathcal{M}, w \Vdash \bot & \text{iff} & w \lhd \emptyset \\ \mathcal{M}, w \Vdash A \vee B & \text{iff} & \exists \alpha.\, w \lhd \alpha \text{ and } \forall v \in \alpha.\, \mathcal{M}, v \Vdash A \text{ or } \mathcal{M}, v \Vdash B \\ \mathcal{M}, w \Vdash \Box A & \text{iff} & \forall w', v.\, w \sqsubseteq w' \text{ and } w' \mathrel{R} v \text{ implies } \mathcal{M}, v \Vdash A \end{array}$$

$$\begin{array}{lll} \mathcal{M}, w \Vdash \cdot & \text{iff} & \text{true} \\ \mathcal{M}, w \Vdash \Gamma, A & \text{iff} & \mathcal{M}, w \Vdash \Gamma \text{ and } \mathcal{M}, w \Vdash A \\ \mathcal{M}, w \Vdash \Gamma, \text{🔒} & \text{iff} & \exists u.\, u \mathrel{R} w \text{ and } \mathcal{M}, u \Vdash \Gamma \end{array}$$

Lemma 1 (Key Lemma). *For any formula A, the truth set $|A|^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \Vdash A\}$ is a localized up-set for an arbitrary cover model $\mathcal{M}$ of $\mathsf{FCK}_\Box$. Similarly, for any context Γ, the truth set $|\Gamma|^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \Vdash \Gamma\}$ is an up-set (that need not be localized).*

Proof. By induction on formula A and context Γ. We use Modal Localization to show that truth sets of modal formulas $\Box A$ are localized, and Lock Refinability to show that truth sets of "locked" contexts $\Gamma, \text{🔒}$ are up-sets. □

Key insight Cover semantics for $\mathsf{FCK}_\Box$ gives us an elegant semantic characterisation of the seemingly ad hoc modification to the ($\bot$-E) and ($\vee$-E) rules. In an attempt to prove completeness by constructing a canonical model (with contexts as worlds) for $\mathsf{FCK}_\Box$, there is a tension in satisfying the Modal Localization rule that is relieved by modifying the ($\bot$-E) and ($\vee$-E) rules to allow an arbitrary Γ' as above. We construct the canonical model by taking contexts for worlds, defining the accessibility relation as $\Gamma \mathrel{R} \Delta$ iff $\exists \Gamma'.\, \Delta = \Gamma, \text{🔒}, \Gamma'$ and $\text{🔒} \notin \Gamma'$, and the covering relation as:

$$\Gamma \lhd \{\Gamma\} \qquad \frac{\Gamma \vdash \bot}{\Gamma, \Gamma' \lhd \emptyset} \qquad \frac{\Gamma \vdash A \vee B \qquad \Gamma, A, \Gamma' \lhd \alpha \qquad \Gamma, B, \Gamma' \lhd \beta}{\Gamma, \Gamma' \lhd \alpha \cup \beta}$$

In contrast to the usual canonical model for IPL, our definition crucially mimics the modified rules to allow an arbitrary context Γ'. In the absence of this modification, Modal Localization fails to hold. A similar relationship can be observed between the modification of the (Hyp) rule and satisfaction of the Lock Refinability condition.

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A Appendix

Full definition of proof rules for $\mathsf{FCK}_{\Box}$

$$\frac{A \in \Gamma \quad \text{🔒} \notin \Gamma'}{\Gamma, \Gamma' \vdash A}\ (\text{Hyp}) \qquad \frac{}{\Gamma \vdash \top}\ (\top\text{-I}) \qquad \frac{\Gamma \vdash \bot}{\Gamma, \Gamma' \vdash A}\ (\bot\text{-E})$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge\text{-I}) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge\text{-E}_1) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge\text{-E}_2)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee\text{-I}_1) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee\text{-I}_2) \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A, \Gamma' \vdash C \quad \Gamma, B, \Gamma' \vdash C}{\Gamma, \Gamma' \vdash C}\ (\vee\text{-E})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ (\Rightarrow\text{-I}) \qquad \frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}\ (\Rightarrow\text{-E}) \qquad \frac{\Gamma, \text{🔒} \vdash A}{\Gamma \vdash \Box A}\ (\Box\text{-I}) \qquad \frac{\Gamma \vdash \Box A \quad \text{🔒} \notin \Gamma'}{\Gamma, \text{🔒}, \Gamma' \vdash B}\ (\Box\text{-E})$$

Full list of conditions on a cover system

- *Local Refinability*: If $w' \sqsupseteq w \lhd \alpha$, then $\exists \alpha'.\, w' \lhd \alpha' \succeq \alpha$.
- *Local Inclusion*: If $w \lhd \alpha$, then $\{w\} \preceq \alpha$
- *Local Identity*: $w \lhd \{w\}$
- *Local Transitivity*: If $w \lhd \alpha$ and $(\forall v \in \alpha.\, \exists \beta_v . v \lhd \beta_v)$, then $w \lhd \bigcup_{v \in \alpha} \beta_v$
- *Modal Localization*: If $w \lhd \alpha \subseteq [R]X$, $w \sqsubseteq w'$ and $w'\ R\ v$, then $\exists \beta.\, v \lhd \beta \subseteq X$.
- *Lock Refinability*: If $w\ R\ v$ and $v \sqsubseteq v'$, then $\exists w'.\, w \sqsubseteq w'\ R\ v'$.

The relation $\preceq\ \subseteq \mathcal{P}(W) \times \mathcal{P}(W)$ in Local Refinability is defined as: $\alpha \preceq \alpha'$ if and only if for all worlds $v' \in \alpha'$ there exists a world $v \in \alpha$ such that $v \sqsubseteq v'$.

Full definition of the satisfaction relation

$$\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & w \in V(p) \\
\mathcal{M}, w \Vdash \top & \text{iff} & \text{true} \\
\mathcal{M}, w \Vdash \bot & \text{iff} & w \lhd \emptyset \\
\mathcal{M}, w \Vdash A \wedge B & \text{iff} & \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B \\
\mathcal{M}, w \Vdash A \vee B & \text{iff} & \exists \alpha.\, w \lhd \alpha \text{ and } \forall v \in \alpha.\, \mathcal{M}, v \Vdash A \text{ or } \mathcal{M}, v \Vdash B \\
\mathcal{M}, w \Vdash A \Rightarrow B & \text{iff} & \forall w'.\, w \sqsubseteq w' \text{ and } \mathcal{M}, w' \Vdash A \text{ implies } \mathcal{M}, w' \Vdash B \\
\mathcal{M}, w \Vdash \Box A & \text{iff} & \forall w', v.\, w \sqsubseteq w' \text{ and } w'\ R\ v \text{ implies } \mathcal{M}, v \Vdash A
\end{array}$$