INTRODUCTION TO TYPES & LAMBDAS

Nachi Valliappan (University of Edinburgh)

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INTRODUCTION TO TYPED FUNCTIONAL PROGRAMMING AND THE LAMBDA CALCULUS

SPONSORED BY

PROOF THEORY

WITHOUT AUTHORITY

HOUSE KEEPING

AUDIENCE OVERVIEW

"PROGRESS" &

"PRESERVATION"

Types (Ty)

$$A, B := b \mid 1 \mid 0 \mid A \times B \mid A + B$$

Contexts (Ctx)

$$\Gamma, \Delta := \cdot \mid \Gamma, x : A \quad (x \text{ not in } \Gamma)$$

VARIABLE HAIKU

NAMES ARE ORNAMENTS
IT IS THE POSITION IN
CONTEXT THAT MATTERS

VARIABLES $\quad \Gamma \vdash_v x : A$

$$\text{ZERO}\ \frac{}{\Gamma, x : A \vdash_v x : A}\ (x \notin \operatorname{dom} \Gamma)$$

$$\text{SUCC}\ \frac{\Gamma \vdash_v x : A}{\Gamma, y : B \vdash_v x : A}\ (y \notin \operatorname{dom} \Gamma)$$

VARIABLES

$$\text{ZERO}\ \frac{}{x : b,\ y : 0 \vdash_v y : 0}\ y \notin \{x\}$$

$$\text{SUCC}\ \frac{x : b,\ y : 0 \vdash_v y : 0}{x : b,\ y : 0,\ z : b \vdash_v y : 0}\ z \notin \{x, y\}$$

NAMED VARIABLES

$$x : b,\ y : 0,\ z : b \ \vdash t : A$$

$$t[u/z,\ v/y,\ w/x]$$

DE BRUIJN INDICES

$$b,\ 0,\ b \ \vdash t : A$$

$$t[u, v, w]$$

TERMS $\Gamma \vdash t : A$

$$\frac{\Gamma \vdash_v x : A}{\Gamma \vdash x : A} \qquad \frac{}{\Gamma \vdash \mathrm{unit} : 1}$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash \mathrm{pair}\ t\ u : A \times B}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathrm{fst}\ t : A} \qquad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathrm{snd}\ t : B}$$

TERMS

$\Gamma \vdash t : A$

$$\frac{\Gamma \vdash t : 0}{\Gamma \vdash \text{abort } t : A}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \text{inl } t : A + B} \qquad \frac{\Gamma \vdash t : B}{\Gamma \vdash \text{inr } t : A + B}$$

$$\frac{\Gamma \vdash s : A + B \qquad \Gamma, x_1 : A \vdash t_1 : C \qquad \Gamma, x_2 : B \vdash t_2 : C}{\Gamma \vdash \text{case } s\ (x_1 . t_1)(x_2 . t_2) : C}$$

DETOURS

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash t : A\end{array} \qquad \begin{array}{c}\vdots \\ \Gamma \vdash u : B\end{array}}{\Gamma \vdash \mathrm{pair}\ t\ u : A \times B}}{\Gamma \vdash \mathrm{fst}\ (\mathrm{pair}\ t\ u) : A}$$

DETOURS

$$
\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash t : A\end{matrix} \qquad \begin{matrix}\vdots \\ \Gamma \vdash u : B\end{matrix}}{\Gamma \vdash \text{pair } t\ u : A \times B}}{\Gamma \vdash \text{fst } (\text{pair } t\ u) : A}
\qquad\qquad
\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash t : A\end{matrix} \qquad \begin{matrix}\vdots \\ \Gamma \vdash u : B\end{matrix}}{\Gamma \vdash \text{pair } t\ u : A \times B}}{\Gamma \vdash \text{snd } (\text{pair } t\ u) : B}
$$

DETOURS

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash t : A\end{matrix} \qquad \begin{matrix}\vdots \\ \Gamma \vdash u : B\end{matrix}}{\Gamma \vdash \mathrm{pair}\ t\ u : A \times B}}{\Gamma \vdash \mathrm{fst}\,(\mathrm{pair}\ t\ u) : A} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash t : A\end{matrix} \qquad \begin{matrix}\vdots \\ \Gamma \vdash u : B\end{matrix}}{\Gamma \vdash \mathrm{pair}\ t\ u : A \times B}}{\Gamma \vdash \mathrm{snd}\,(\mathrm{pair}\ t\ u) : B}$$

$(\beta_{\times_1})$ $\mathrm{fst}\,(\mathrm{pair}\ t\ u) = t$ $\qquad$ $(\beta_{\times_2})$ $\mathrm{snd}\,(\mathrm{pair}\ t\ u) = u$

DETOURS

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash t : A\end{matrix} \qquad \begin{matrix}\vdots \\ \Gamma \vdash u : B\end{matrix}}{\Gamma \vdash \mathrm{pair}\ t\ u : A \times B}}{\Gamma \vdash \mathrm{fst}\,(\mathrm{pair}\ t\ u) : A} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash t : A\end{matrix} \qquad \begin{matrix}\vdots \\ \Gamma \vdash u : B\end{matrix}}{\Gamma \vdash \mathrm{pair}\ t\ u : A \times B}}{\Gamma \vdash \mathrm{snd}\,(\mathrm{pair}\ t\ u) : B}$$

$(\beta_{\times_1})$ $\mathrm{fst}\,(\mathrm{pair}\ t\ u) = t$ $\qquad$ $(\beta_{\times_2})$ $\mathrm{snd}\,(\mathrm{pair}\ t\ u) = u$

$(\eta_{\times})$ For $\Gamma \vdash t : A \times B$, $\ t = \mathrm{pair}\ (\mathrm{fst}\ t)\ (\mathrm{snd}\ t)$

DETOURS

$$
\dfrac{\begin{matrix} \vdots \\ \Gamma \vdash t : A+B \end{matrix} \qquad \dfrac{\Gamma, x_1 : A \vdash x_1 : A}{\Gamma, x_1 : A \vdash \mathrm{inl}\, x_1 : A+B} \qquad \dfrac{\Gamma, x_2 : B \vdash x_2 : B}{\Gamma, x_2 : B \vdash \mathrm{inr}\, x_2 : A+B}}{\Gamma \vdash \mathrm{case}\ t\ (x_1 . \mathrm{inl}\, x_1)\ (x_2 . \mathrm{inr}\, x_2) : A+B}
$$

(η_+) For $\Gamma \vdash t : A+B$,

$$t = \mathrm{case}\ t\ (x_1 . \mathrm{inl}\, x_1)\ (x_2 . \mathrm{inr}\, x_2)$$

DETOURS

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash s : A\end{matrix}}{\Gamma \vdash \mathrm{inl}\ s : A+B} \qquad \begin{matrix}\vdots \\ \Gamma, x_1 : A \vdash t_1 : C\end{matrix} \qquad \begin{matrix}\vdots \\ \Gamma, x_2 : B \vdash t_2 : C\end{matrix}}{\Gamma \vdash \mathrm{case}\ (\mathrm{inl}\ s)\ (x_1 . t_1)\ (x_2 . t_2) : C}$$

$$(\beta_{+1}) \quad \mathrm{case}\ (\mathrm{inl}\ s)\ (x_1 . t_1)\ (x_2 . t_2) = t_1[s/x_1]$$

$$(\beta_{+2}) \quad \mathrm{case}\ (\mathrm{inr}\ s)\ (x_1 . t_1)\ (x_2 . t_2) = t_2[s/x_2]$$

SUBSTITUTION*

$$t\,[s/x]$$

Given $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$
we have $\Gamma \vdash t\,[s/x] : B$

e.g. $\text{pair}\ x\ y\ [\text{unit}/x] \equiv \text{pair unit}\ y$

SUBSTITUTION*

$t[s/x]$

Given $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$
we have $\Gamma \vdash t[s/x] : B$

e.g. $\text{pair}\ x\ y\ [\text{unit}/x] \equiv \text{pair unit}\ y$

non e.g. $\text{case}\ s\ (x_1 . x)\ (x_2 . x)\ [x_1/x]$

$\equiv \text{case}\ s\ (x_1 . x_1)\ (x_2 . x_1)\ [x_1/x]$

WEAKENING*

$\uparrow t$

Given $\Gamma \vdash t : A$ and $\Gamma \subseteq \Gamma'$

we have $\Gamma' \vdash \uparrow t : A$

e.g. Given $x : b, y : 1 \vdash t : b+1$

we have $x : b, y : 1, z : 0 \vdash \uparrow t : b+1$

also have $x : b, z : 0, y : 1 \vdash \uparrow t : b+1$

DETOURS

$\beta_{\times 1}, \beta_{\times 2}, \beta_{+1}, \beta_{+2}$

$\eta_{\times}, \eta_{+}$

(η_1) For $\Gamma \vdash t : 1$, $t = \text{unit}$

(η_0) For $\Gamma \vdash t : 0$, $t = \text{abort } t$

DETOURS

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash x : A+B\end{array} \quad \begin{array}{c}\vdots\\ \Gamma, x_1 : A \vdash \mathrm{pair}\ t_1\ u_1 : C \times D\end{array} \quad \begin{array}{c}\vdots\\ \Gamma, x_2 : B \vdash \mathrm{pair}\ t_2\ u_2 : C \times D\end{array}}{\Gamma \vdash \mathrm{case}\ x\ (x_1 . \mathrm{pair}\ t_1\ u_1)\ (x_2 . \mathrm{pair}\ t_2\ u_2) : C \times D}}{\Gamma \vdash \mathrm{fst}(\mathrm{case}\ x\ (x_1 . \mathrm{pair}\ t_1\ u_1)\ (x_2 . \mathrm{pair}\ t_2\ u_2)) : C}$$

DETOURS

$$
\dfrac{\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash x : A+B\end{array} \quad \begin{array}{c}\vdots \\ \Gamma, x_1 : A \vdash \mathrm{pair}\ t_1\ u_1 : C \times D\end{array} \quad \begin{array}{c}\vdots \\ \Gamma, x_2 : B \vdash \mathrm{pair}\ t_2\ u_2 : C \times D\end{array}}{\Gamma \vdash \mathrm{case}\ x\ (x_1.\ \mathrm{pair}\ t_1\ u_1)\ (x_2.\ \mathrm{pair}\ t_2\ u_2) : C \times D}}{\Gamma \vdash \mathrm{fst}(\mathrm{case}\ x\ (x_1.\ \mathrm{pair}\ t_1\ u_1)\ (x_2.\ \mathrm{pair}\ t_2\ u_2)) : C}
$$

$$
\begin{aligned}
\pi_{+}^{\mathrm{fst}} \quad & \mathrm{fst}\ (\mathrm{case}\ s\ (x_1.\ t)\ (x_2.\ u)) \\
& = (\mathrm{case}\ s\ (x_1.\ \mathrm{fst}\ t)\ (x_2.\ \mathrm{fst}\ u))
\end{aligned}
$$

DETOURS

$$\pi_{+}^{E} \quad E[\text{case } x\ (x_1.\, t_1)\ (x_2.\, t_2)]$$

$$= \text{case } x\ (x_1.\, E[t_1])\ (x_2.\, E[t_2])$$

$$E ::= \text{Fst}[\,] \mid \text{snd}[\,] \mid \text{abort}[\,]$$
$$\mid \text{case}[\,]\ (x_1.\, t_1)\ (x_2.\, t_2)$$

DETOURS

$$\pi_+^E \quad E[\text{case } x\ (x_1.\, t_1)\ (x_2.\, t_2)]$$

$$= \text{case } x\ (x_1.\, E[t_1])\ (x_2.\, E[t_2])$$

$$E ::= \text{Fst}[\,] \mid \text{Snd}[\,] \mid \text{abort}[\,]$$

$$\mid \text{case}[\,]\ (x_1.\, t_1)\ (x_2.\, t_2)$$

$$\pi_0^E \quad E[\text{abort } t] = \text{abort}(E[t])$$

DETOURS

$\beta_{\times 1}, \beta_{\times 2}, \beta_{+1}, \beta_{+2}$] Reduction

$\eta_{\times}, \eta_{+}, \eta_{1}, \eta_{0}$] Expansion

π^{E}_{+}, π^{E}_{0}] Hoisting

$\ldots (\text{e.g. } \pi^{I}_{+}, \pi^{I}_{0}) \ldots$] Categorical semantics
Dead code elimination
...

EXERCISE

* Show Π_{+}^{pair} is admissible

i.e. $\mathrm{pair}\,(\mathrm{case}\ s\ (x_1.t_1)\ (x_2.t_2))$
$(\mathrm{case}\ s\ (x_1.u_1)\ (x_2.u_2))$
$= \mathrm{case}\ s\ (x_1.\,\mathrm{pair}\ t_1\,u_1)\ (x_2.\,\mathrm{pair}\ t_2\,u_2)$

* Show Π_{0}^{pair} is admissible.

* Is Π_{+}^{inl} or Π_{0}^{inl} admissible?

INTERPRETING TYPES & CONTEXTS

$$[\![b]\!] = V_b$$

$$[\![0]\!] = \emptyset \qquad [\![\cdot]\!] = \{()\}$$

$$[\![1]\!] = \{()\} \qquad [\![\Gamma, x : A]\!] = [\![\Gamma]\!] \times [\![A]\!]$$

$$[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$$

$$[\![A + B]\!] = [\![A]\!] \uplus [\![B]\!]$$

Interpreting Terms

$$\mathrm{Tm}\,\Gamma\,A = \{ t \mid \Gamma \vdash t : A \}$$

$$[\![\_]\!] : \overbrace{\mathrm{Tm}\,\Gamma\,A} \to [\![\Gamma]\!] \to [\![A]\!]$$

$$[\![x]\!]\ \gamma = \gamma(x)$$

$$[\![\mathrm{unit}]\!]\ \gamma = ()$$

$$[\![\mathrm{abort}\ t]\!]\ \gamma = \mathrm{empty}(\overbrace{[\![t]\!]\ \gamma}^{[\![t]\!] \in \emptyset})$$

$$\mathrm{empty} : \emptyset \to [\![A]\!]$$

INTERPRETING TERMS

$$[\![\_]\!] : \mathrm{Tm}\ \Gamma\ A \longrightarrow [\![\Gamma]\!] \to [\![A]\!]$$

...

$$[\![\mathrm{pair}\ t\ u]\!]\ \gamma = (\overbrace{[\![t]\!]\ \gamma}^{[\![A_1]\!]}, \overbrace{[\![u]\!]\ \gamma}^{[\![A_2]\!]})$$

$$[\![\mathrm{fst}\ t]\!]\ \gamma = \mathrm{proj}_1 ([\![t]\!]\ \gamma)$$

$$[\![\mathrm{snd}\ t]\!]\ \gamma = \mathrm{proj}_2 (\underbrace{[\![t]\!]\ \gamma}_{[\![A_1]\!] \times [\![A_2]\!]})$$

INTERPRETING TERMS

$$[\![\_]\!] : \mathrm{Tm}\ \Gamma\ A \to [\![\Gamma]\!] \to [\![A]\!]$$

...

$$[\![\mathrm{inl}\ t]\!]\ \gamma = (1, [\![t]\!]\ \gamma)$$

$$[\![\mathrm{inr}\ t]\!]\ \gamma = (2, [\![t]\!]\ \gamma)$$

$$\left[\!\!\left[\begin{array}{l} \mathrm{case}\ s \\ \quad (x_1 . t_1) \\ \quad (x_2 . t_2) \end{array} \right]\!\!\right] \gamma = \begin{cases} [\![t_1]\!]\ (\gamma, v_1) & \text{if } [\![s]\!]\ \gamma = (1, v_1) \\ [\![t_2]\!]\ (\gamma, v_2) & \text{if } [\![s]\!]\ \gamma = (2, v_2) \end{cases}$$

EXAMPLE

$\mathrm{Bool} = 1+1 \qquad \mathrm{true} = \mathrm{inl}\,() \qquad \mathrm{false} = \mathrm{inr}\,()$

$[\![\mathrm{true}]\!]\gamma = (1, ()) \qquad [\![\mathrm{false}]\!]\gamma = (2, ())$

$\mathrm{not}\ x = \mathrm{case}\ x\ (\_.\,\mathrm{false})\ (\_.\,\mathrm{true})$

"macro"

EXAMPLE

Interpret $\cdot, x : \mathrm{Bool} \vdash \mathrm{not}\ x : \mathrm{Bool}$

with $((), (1, ())) : [\![\cdot, x : \mathrm{Bool}]\!]$

$$[\![\mathrm{not}\ x]\!]((), (1, ())) \quad : [\![\mathrm{Bool}]\!]$$

$= \quad \ldots$

$= \quad \ldots$

$= (2, ())$

EXAMPLE

Interpret $\cdot, x : \mathrm{Bool} \vdash \mathrm{not}\ x : \mathrm{Bool}$

with $((), (1, ())) : [\![\cdot, x : \mathrm{Bool}]\!]$

$$[\![\mathrm{not}\ x]\!] ((), (1, ())) \quad : [\![\mathrm{Bool}]\!]$$

$$= [\![\mathrm{case}\ x\ (x_1.\ \mathrm{false})\ (x_2.\ \mathrm{true})]\!] ((), \mathrm{true})$$

$$= [\![\mathrm{false}]\!] ((), (1, ()), ())$$

since $[\![x]\!] ((), \mathrm{true}) = (1, ())$

$$= (2, ())$$

Values & Environments

$$\frac{v \in \mathcal{V}_b}{\vdash_{VAL} v : b} \qquad \frac{}{\vdash_{VAL} \mathrm{unit} : 1}$$

$$\frac{\vdash_{VAL} v_1 : A \qquad \vdash_{VAL} v_2 : B}{\vdash_{VAL} \mathrm{pair}\ v_1\, v_2 : A \times B}$$

$$\frac{\vdash_{VAL} v : A}{\vdash_{VAL} \mathrm{inl}\ v : A + B} \qquad \frac{\vdash_{VAL} v : B}{\vdash_{VAL} \mathrm{inr}\ v : A + B}$$

VALUES & ENVIRONMENTS

$$\frac{}{\vdash_{\text{ENV}} \text{nil} : \bullet} \qquad \frac{\vdash_{\text{ENV}} \gamma : \Gamma \qquad \vdash_{\text{VAL}} v : A}{\vdash_{\text{ENV}} \text{Cons}\ \gamma\ v : \Gamma, A}$$

$$\text{Val}\ A = \{ v \mid \vdash_{\text{VAL}} v : A \}$$

$$\text{Env}\ \Gamma = \{ \gamma \mid \vdash_{\text{ENV}} \gamma : \Gamma \}$$

EVALUATION

$$\mathrm{eval} : \mathrm{Tm}\ \Gamma\ A \to \mathrm{Env}\ \Gamma \to \mathrm{Val}\ A$$

...

$$\mathrm{eval}\ \mathrm{unit}\ \gamma = \mathrm{unit}$$

$$\mathrm{eval}\ (\mathrm{pair}\ t\ u)\ \gamma = \mathrm{pair}\ (\mathrm{eval}\ t\ \gamma)\ (\mathrm{eval}\ u\ \gamma)$$

...

EVALUATION

$$
\begin{tikzcd}
\llbracket \_ \rrbracket : \mathrm{Tm}\,\Gamma\,A \arrow[r] & \llbracket \Gamma \rrbracket \arrow[r] & \llbracket A \rrbracket \arrow[d, "\mathrm{reify}"] \\
\mathrm{eval} : \mathrm{Tm}\,\Gamma\,A \arrow[r] & \mathrm{Env}\,\Gamma \arrow[r] \arrow[u, "\mathrm{reflect}"] & \mathrm{Val}\,A
\end{tikzcd}
$$

REFLECTION

$$\mathit{reflect}_A : \mathrm{Val}\ A \to [\![A]\!]$$

$$\mathit{reflect}_b \quad v \quad = \quad v$$

$$\mathit{reflect}_1 \quad \mathrm{unit} \quad = \quad ()$$

$$\mathit{reflect}_{A_1 \times A_2} (\mathrm{pair}\ v_1\ v_2) = (\mathit{reflect}_{A_1} v_1 , \mathit{reflect}_{A_2} v_2)$$

$$\mathit{reflect}_{A_1 + A_2} (\mathrm{inl}\ v) = (1, \mathit{reflect}_{A_1} v)$$

$$\mathit{reflect}_{A_1 + A_2} (\mathrm{inr}\ v) = (2, \mathit{reflect}_{A_2} v)$$

REFLECTION

$$\mathit{reflect}_\Gamma : \mathrm{Env}\,\Gamma \to [\![\Gamma]\!]$$

$$\mathit{reflect}_\bullet \quad \mathit{unit} \quad = \quad ()$$

$$\mathit{reflect}_{(\Gamma, x:A)} \,(\mathit{cons}\ \gamma\ V) = (\mathit{reflect}_\Gamma\ \gamma\ ,\ \mathit{reflect}_A\ V\ )$$

REIFICATION

$$\mathrm{reify}_A : [\![A]\!] \to \mathrm{Val}\ A$$

$$\mathrm{reify}_b \quad v = v$$

$$\mathrm{reify}_1 \quad () = \mathrm{unit}$$

$$\mathrm{reify}_{A_1 \times A_2} \ (v_1, v_2) = \mathrm{pair}\ (\mathrm{reify}_{A_1}\ v_1)\ (\mathrm{reify}_{A_2}\ v_2)$$

$$\mathrm{reify}_{A_1 + A_2} \ (1, v_1) = \mathrm{inl}\ (\mathrm{reify}_{A_1}\ v)$$

$$\mathrm{reify}_{A_1 + A_2} \ (2, v_2) = \mathrm{inr}\ (\mathrm{reify}_{A_2}\ v)$$

EVALUATION

$$\mathrm{eval} : \mathrm{Tm}\ \Gamma\ A \to \mathrm{Env}\ \Gamma \to \mathrm{Val}\ A$$

$$\mathrm{eval}\ t\ \gamma = \mathrm{reify}\,\Big(\underbrace{[\![t]\!]\,\big(\overbrace{\mathrm{reflect}\ \gamma}^{[\![\Gamma]\!]}\big)}_{[\![A]\!]}\Big)$$

EVALUATION

$$\mathrm{eval} : \mathrm{Tm}\ \Gamma\ A \to \mathrm{Env}\ \Gamma \to \mathrm{Val}\ A$$

$$\mathrm{eval}\ t\ \gamma = \mathrm{reify}\ (\underbrace{[\![t]\!]\ (\overbrace{\mathrm{reflect}\ \gamma}^{[\![\Gamma]\!]})}_{[\![A]\!]})$$

e.g. $\mathrm{eval}\ (\mathrm{not}\ \mathrm{true})\ \mathrm{nil} = \mathrm{false}$

Open Evaluation

$$
\begin{tikzcd}
\llbracket \_ \rrbracket : \mathrm{Tm}\,\Gamma\,A \arrow[r] & \llbracket \Gamma \rrbracket_\Delta \arrow[r] & \llbracket A \rrbracket_\Delta \arrow[d, "\text{reify}"] \\
\mathrm{eval} : \mathrm{Tm}\,\Gamma\,A \arrow[r] & \mathrm{Env}\,\Delta\,\Gamma \arrow[u, "\text{reflect}"] \arrow[r] & \mathrm{Val}\,\Delta\,A
\end{tikzcd}
$$