INTRODUCTION TO TYPES & LAMBDAS

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OPEN EVALUATION

$$
\begin{tikzcd}
\llbracket \_ \rrbracket : \mathrm{Tm}\,\Gamma\,A \arrow[r] & \llbracket \Gamma \rrbracket_\Delta \arrow[r] & \llbracket A \rrbracket_\Delta \arrow[d, "\mathrm{reify}"] \\
\mathrm{eval} : \mathrm{Tm}\,\Gamma\,A \arrow[r] & \mathrm{Env}\,\Delta\,\Gamma \arrow[r] \arrow[u, "\mathrm{reflect}"] & \mathrm{Val}\,\Delta\,A
\end{tikzcd}
$$

PREVIOUSLY

$$\begin{tikzcd}
\llbracket A \rrbracket \arrow[d, shift left, "\text{reify}"] \\
\mathrm{Val}\ A \arrow[u, shift left, "\text{reflect}"]
\end{tikzcd}$$

TODAY

$$\begin{tikzcd}
& \llbracket A \rrbracket_{\Delta} \arrow[dr, "\text{reify}"] & \\
\mathrm{NE}\ \Delta\ A \arrow[ur, "\text{reflect}"] \arrow[rr, "\text{emb}"] & & \mathrm{NF}\ \Delta\ A
\end{tikzcd}$$

SUBFORMULAS

Formula : 0

Subformulas : $\{0\}$

Formula : $0+1$

Subformulas : $\{0, 1, 0+1\}$

Formula : $(b \times 0)+1$

Subformulas : $\{\, b, 0, 1, b \times 0, (b \times 0)+1 \,\}$

SUBFORMULAS

"occurrence"

write $A \trianglelefteq B$

when A is a subformula of B

write $A \trianglelefteq \Gamma$

when for some $B \in \mathrm{cod}(\Gamma)$, $A \trianglelefteq B$

NEUTRAL TERMS

$$\frac{\Gamma \vdash_{V} x : A}{\Gamma \vdash_{NE} x : A}$$

$$\frac{\Gamma \vdash_{NE} t : A \times B}{\Gamma \vdash_{NE} \mathsf{fst}\ t : A} \qquad \frac{\Gamma \vdash_{NE} t : A \times B}{\Gamma \vdash_{NE} \mathsf{snd}\ t : B}$$

NEUTRAL TERMS

$$\frac{\Gamma \vdash_{V} x : A}{\Gamma \vdash_{NE} x : A}$$

$$\frac{\Gamma \vdash_{NE} t : A \times B}{\Gamma \vdash_{NE} \mathsf{fst}\ t : A} \qquad \frac{\Gamma \vdash_{NE} t : A \times B}{\Gamma \vdash_{NE} \mathsf{snd}\ t : B}$$

Observation : If $\Gamma \vdash_{NE} t : A$, then $A \triangleleft \Gamma$

NORMAL FORMS

$$\frac{\Gamma \vdash_{NE} t : b}{\Gamma \vdash_{NF} t : b} \qquad \frac{v \in V_b}{\Gamma \vdash_{NF} v : b} \qquad \Gamma \vdash_{NF} \mathrm{unit} : 1$$

$$\frac{\Gamma \vdash_{NE} t : 0}{\Gamma \vdash_{NF} \mathrm{abort}\ t : A} \qquad \frac{\Gamma \vdash_{NF} t : A \qquad \Gamma \vdash_{NF} u : B}{\Gamma \vdash_{NF} \mathrm{pair}\ t\ u : A \times B}$$

NORMAL FORMS

$$\frac{\Gamma \vdash_{NE} t : b}{\Gamma \vdash_{NF} t : b} \qquad \frac{v \in V_b}{\Gamma \vdash_{NF} v : b} \qquad \Gamma \vdash_{NF} \mathrm{unit} : 1$$

$$\frac{\Gamma \vdash_{NE} t : 0}{\Gamma \vdash_{NF} \mathrm{abort}\ t : A} \qquad \frac{\Gamma \vdash_{NF} t : A \qquad \Gamma \vdash_{NF} u : B}{\Gamma \vdash_{NF} \mathrm{pair}\ t\ u : A \times B}$$

Observation : Given $\Gamma \vdash_{NF} t : A$,

and any subterm $\Gamma' \vdash_{NE/NF} u : A'$,

$\forall B \in \{A'\} \cup \mathrm{cod}(\Gamma) .\ B \triangleleft A$ or $B \triangleleft \Gamma$

"SUBFORMULA PROPERTY"

NORMAL FORMS

$$\frac{\Gamma \vdash_{NF} t : A}{\Gamma \vdash_{NF} \mathrm{inl}\ t : A + B} \qquad \frac{\Gamma \vdash_{NF} t : B}{\Gamma \vdash_{NF} \mathrm{inr}\ t : A + B}$$

$$\frac{\Gamma \vdash_{NE} s : A + B \quad \Gamma, x_1 : A \vdash_{NF} t_1 : C \quad \Gamma, x_2 : B \vdash_{NF} t_2 : C}{\Gamma \vdash_{NF} \mathrm{case}\ s\ (x_1 . t_1)\ (x_2 . t_2) : C}$$

DETOURS NO MORE!

EXERCISE

* Are you able to construct any of the detours from yesterday in normal form syntax?
* Is $x : \mathrm{Bool} \vdash x : \mathrm{Bool}$ in normal form?
* Define a function $\mathrm{emb} : \mathrm{NE}\,\Delta\,A \longrightarrow \mathrm{NF}\,\Delta\,A$
* What does emb do, equationally?

  i.e., show $n = \mathrm{emb}\ n$ in the eq. theory.

Environments

$$\frac{}{\Delta \vdash_{ENV} nil : \cdot} \qquad \frac{\Delta \vdash_{ENV} \gamma : \Gamma \quad \Delta \vdash_{NE} t : A}{\Delta \vdash_{ENV} \mathrm{Cons}\ \gamma\ t : \Gamma, x : A}\ x \notin \mathrm{dom}\,\Gamma$$

$$\mathrm{Env}\ \Delta\ \Gamma = \{ \gamma \mid \Delta \vdash_{ENV} \gamma : \Gamma \}$$

$$\mathrm{NF}\ \Delta\ A = \{ t \mid \Delta \vdash_{NF} t : A \} \qquad \mathrm{NE}\ \Delta\ A = \{ t \mid \Delta \vdash_{NE} t : A \}$$

EXERCISE: Convince yourself that for any $\Delta \in \mathrm{Ctx}$ there is a δ s.t. $\Delta \vdash_{ENV} \delta : \Delta$

Previously $[\![\_]\!] : Ty \to Set$

Now $[\![\_]\!] : Ty \to Ctx \to Set$

NOTATION

family of sets

$X : Ctx \to Set$

indexing set

indexed as X_Δ, instead of $X\,\Delta$

defined as $\{\ldots\Delta\ldots\}_{\Delta \in Ctx}$, instead of $\lambda\Delta.\ldots\Delta\ldots$

Interpreting Types & Contexts

$$[\![b]\!]_\Delta = \mathrm{NF}\ \Delta\ b$$

$$[\![0]\!]_\Delta = \emptyset$$

$$[\![1]\!]_\Delta = \{ () \}$$

$$[\![A \times B]\!]_\Delta = [\![A]\!]_\Delta \times [\![B]\!]_\Delta$$

$$[\![A + B]\!]_\Delta = [\![A]\!]_\Delta \uplus [\![B]\!]_\Delta$$

$$[\![\cdot]\!]_\Delta = \{ () \}$$

$$[\![\Gamma, x : A]\!]_\Delta = [\![\Gamma]\!]_\Delta \times [\![A]\!]_\Delta$$

Interpreting Terms

$$[\![\_]\!] : \mathrm{Tm}\ \Gamma\ A \to \forall \Delta.\ [\![\Gamma]\!]_\Delta \to [\![A]\!]_\Delta$$

...

$$[\![\mathrm{pair}\ t\ u]\!]\ \gamma = ([\![t]\!]\ \gamma , [\![u]\!]\ \gamma)$$

$$[\![\mathrm{fst}\ t]\!]\ \gamma = \mathrm{proj}_1 ([\![t]\!]\ \gamma)$$

$$[\![\mathrm{snd}\ t]\!]\ \gamma = \mathrm{proj}_2 ([\![t]\!]\ \gamma)$$

...

REIFICATION

$$\mathrm{reify}_A : \forall \Delta.\ [\![A]\!]_\Delta \longrightarrow \mathrm{NF}\ \Delta\ A$$

$$\mathrm{reify}_b \quad t \;=\; t$$

$$\mathrm{reify}_1 \quad () \;=\; \mathrm{unit}$$

$$\mathrm{reify}_{A_1 \times A_2} \ (v_1, v_2) \;=\; \mathrm{pair}\ (\mathrm{reify}_{A_1}\ v_1)\ (\mathrm{reify}_{A_2}\ v_2)$$

$$\mathrm{reify}_{A_1 + A_2} \ (1, v_1) \;=\; \mathrm{inl}\ (\mathrm{reify}_{A_1}\ v)$$

$$\mathrm{reify}_{A_1 + A_2} \ (2, v_2) \;=\; \mathrm{inr}\ (\mathrm{reify}_{A_2}\ v)$$

REFLECTION

$$reflect_A : \forall \Delta . \, NE \, \Delta \, A \to [\![A]\!]_\Delta$$

$$reflect_b \quad t \quad = \quad t$$

$$reflect_1 \quad n \quad = \quad ()$$

$$reflect_0 \quad n \quad = \quad ??$$

$$reflect_{A_1 \times A_2} \quad n \quad = \quad (reflect_{A_1} (fst \; n), reflect_{A_2} (snd \; n))$$

$$reflect_{A_1 + A_2} \quad n \quad = \quad ??$$

INTERPRETING TYPES

$$[\![b]\!]_\Delta = \mathrm{NF}\ \Delta\ b$$

$$[\![0]\!]_\Delta = ??$$

$$[\![1]\!]_\Delta = \{ () \}$$

$$[\![A \times B]\!]_\Delta = [\![A]\!]_\Delta \times [\![B]\!]_\Delta$$

$$[\![A + B]\!]_\Delta = ??$$

Recollect

$$\frac{\Gamma \vdash_{NE} t : 0}{\Gamma \vdash_{NF} \text{abort } t : A}$$

$$\frac{\Gamma \vdash_{NE} s : A + B \qquad \Gamma, x_1 : A \vdash_{NF} t_1 : C \qquad \Gamma, x_2 : B \vdash_{NF} t_2 : C}{\Gamma \vdash_{NF} \text{case } s\ (x_1 . t_1)(x_2 . t_2) : C}$$

COVERING FAMILY / SET

Define $\mathcal{C} : (Ctx \to Set) \to (Ctx \to Set)$ as:

for any $X : Ctx \to Set$, $\Delta \in Ctx$, define $\mathcal{C}X_\Delta$ inductively by

$$\frac{x \in X_\Delta}{\mathrm{val}\ x \in \mathcal{C}X_\Delta} \qquad \frac{t \in NE\ \Delta\ 0}{\mathrm{abort}\ t \in \mathcal{C}X_\Delta}$$

$$\frac{t \in NE\ \Delta\ (A_1 + A_2) \qquad c_1 \in \mathcal{C}X_{\Delta, x_1 : A_1} \qquad c \in \mathcal{C}X_{\Delta, x_2 : A_2}}{\mathrm{case}\ s\ c_1\ c_2 \in \mathcal{C}X_\Delta}$$

INTERPRETING TYPES

$$\dot{\emptyset} = \{\emptyset\}_{\Delta \in Ctx}$$

$$X \mathbin{\dot{\uplus}} Y = \{X_\Delta \uplus Y_\Delta\}_{\Delta \in Ctx}$$

$$[\![b]\!]_\Delta = NF\ \Delta\ b$$

$$[\![0]\!]_\Delta = \mathcal{C}\,\dot{\emptyset}_\Delta$$

$$[\![1]\!]_\Delta = \{()\}$$

$$[\![A \times B]\!]_\Delta = [\![A]\!]_\Delta \times [\![B]\!]_\Delta$$

$$[\![A + B]\!]_\Delta = \mathcal{C}([\![A]\!] \mathbin{\dot{\uplus}} [\![B]\!])_\Delta$$

Interpreting Types & Contexts

$$[\![b]\!]_\Delta = \mathrm{NF}\,\Delta\,b$$

$$[\![0]\!]_\Delta = \mathcal{C}\dot{\emptyset}_\Delta \qquad [\![\cdot]\!]_\Delta = \{()\}$$

$$[\![1]\!]_\Delta = \{()\} \qquad [\![\Gamma, x : A]\!]_\Delta = [\![\Gamma]\!]_\Delta \times [\![A]\!]_\Delta$$

$$[\![A \times B]\!]_\Delta = [\![A]\!]_\Delta \times [\![B]\!]_\Delta$$

$$[\![A + B]\!]_\Delta = \mathcal{C}([\![A]\!] \dot{\uplus} [\![B]\!])_\Delta$$

REFLECTION

$$\mathrm{reflect}_A : \forall \Delta.\ \mathrm{NE}\ \Delta\ A \to [\![A]\!]_\Delta$$

...

$$\mathrm{reflect}_0 \quad n \quad = \mathrm{abort}\ n$$

$$\mathrm{reflect}_{A_1 + A_2} \quad n \quad = \mathrm{case}\ n$$

$$(\mathrm{val}\ \overbrace{(\mathrm{reflect}_{A_1}\ x_1)}^{[\![A_1]\!]_{\Delta, x_1 : A_1}}) : \mathcal{C}\, [\![A_1]\!]_{\Delta, x_1 : A_1}$$

$$(\mathrm{val}\ \underbrace{(\mathrm{reflect}_{A_2}\ x_2)}_{[\![A_2]\!]_{\Delta, x_2 : A_2}}) : \mathcal{C}\, [\![A_2]\!]_{\Delta, x_2 : A_2}$$

SET	FAMILY OF SETS	(alternative notation)
$X : \mathrm{Set}$	$X : \mathrm{Ctx} \to \mathrm{Set}$	$\{X_\Delta\}_{\Delta \in \mathrm{Ctx}}$
$X \longrightarrow Y$	$X \mathrel{\dot{\to}} Y \triangleq \forall \Delta.\, X_\Delta \to Y_\Delta$	$\{X_\Delta \to Y_\Delta\}_{\Delta \in \mathrm{Ctx}}$
$X \times Y$	$X \mathbin{\dot{\times}} Y \triangleq \forall \Delta.\, X_\Delta \times Y_\Delta$	$\{X_\Delta \times Y_\Delta\}_{\Delta \in \mathrm{Ctx}}$
$X \uplus Y$	$X \mathbin{\dot{\uplus}} Y \triangleq \forall \Delta.\, X_\Delta \uplus Y_\Delta$	$\{X_\Delta \uplus Y_\Delta\}_{\Delta \in \mathrm{Ctx}}$
$\emptyset$	$\dot{\emptyset} \triangleq \forall \Delta.\, \emptyset$	$\{\emptyset\}_{\Delta \in \mathrm{Ctx}}$
$\top = \{()\}$	$\dot{\top} \triangleq \forall \Delta.\, \{()\}$	$\{\{()\}\}_{\Delta \in \mathrm{Ctx}}$

Interpreting Types & Contexts

$$\dot{\mathrm{NF}}\, A \triangleq \{ t \mid \Delta \vdash_{\mathrm{NF}} t : A \}_{\Delta \in \mathrm{Ctx}}$$

$$[\![b]\!] = \dot{\mathrm{NF}}\, b$$

$$[\![0]\!] = \mathcal{C}\, \dot{\emptyset}$$

$$[\![\cdot]\!] = \dot{\top}$$

$$[\![1]\!] = \dot{\top}$$

$$[\![\Gamma, x : A]\!] = [\![\Gamma]\!] \mathbin{\dot{\times}} [\![A]\!]$$

$$[\![A \times B]\!] = [\![A]\!] \mathbin{\dot{\times}} [\![B]\!]$$

$$[\![A + B]\!] = \mathcal{C}\left([\![A]\!] \mathbin{\dot{\uplus}} [\![B]\!] \right)$$

OPERATIONS ON COVERING FAMILIES

$$fmap : X \dot{\to} Y \to \mathcal{C}X \dot{\to} \mathcal{C}Y$$

$$return : X \dot{\to} \mathcal{C}X$$

$$join : \mathcal{C}(\mathcal{C}X) \dot{\to} \mathcal{C}X$$

$$strength : [\![\Gamma]\!] \dot{\times} \mathcal{C}X \dot{\to} \mathcal{C}([\![\Gamma]\!] \dot{\times} X)$$

$$loc_{\dot{N}FA} : \mathcal{C}(\dot{N}FA) \dot{\to} \dot{N}FA$$

$$loc_{[\![A]\!]} : \mathcal{C}[\![A]\!] \dot{\to} [\![A]\!]$$

LOCALISATION

$$\mathrm{loc}_{\dot{NFA}} : \mathcal{C}(\dot{NFA}) \xrightarrow{\cdot} \dot{NFA}$$

$$\mathrm{val}\ t \xrightarrow{\mathrm{loc}} t$$

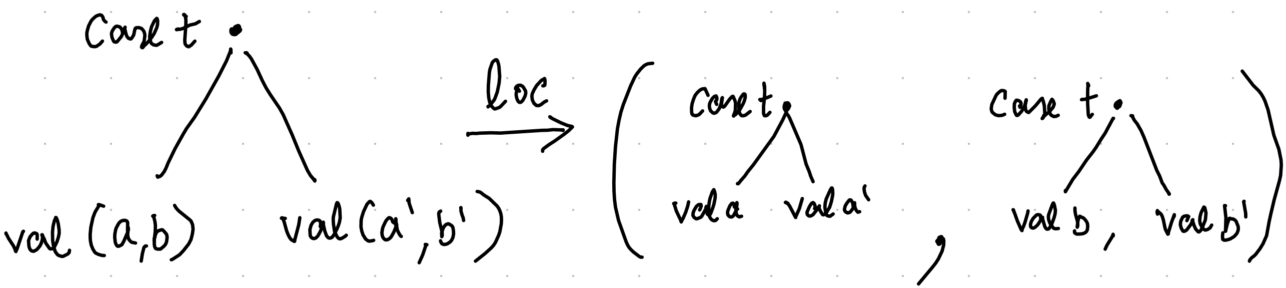
$$\mathrm{abort}\ t \xrightarrow{\mathrm{loc}} \mathrm{abort}\ t$$

$$\begin{tikzcd}
 & \mathrm{case}\ t \arrow[dl, dash] \arrow[dr, dash] & & & & \mathrm{case}\ t \arrow[dl, dash] \arrow[dr, dash] & \\
c_1 & & c_2 \arrow[rr, phantom, "\xrightarrow{\mathrm{loc}}"] & & \mathrm{loc}\ c_1 & & \mathrm{loc}\ c_2
\end{tikzcd}$$

LOCALISATION

$$
\begin{tikzcd}
\mathrm{loc}_{[\![A \times B]\!]} : \mathcal{C}([\![A]\!] \times [\![B]\!]) \arrow[r] \arrow[dr] & [\![A]\!] \times [\![B]\!] \\
& \mathcal{C}[\![A]\!] \times \mathcal{C}[\![B]\!] \arrow[u, "\mathrm{loc}_{[\![A]\!]} \times \mathrm{loc}_{[\![B]\!]}"']
\end{tikzcd}
$$

as an example,



Reification

$$\mathrm{reify}_0 : \mathcal{C}\,\dot{\emptyset}_\Delta \longrightarrow \mathrm{NF}\,\Delta\,0$$

$$\mathrm{reify}_0\ c \;=\; \mathrm{loc}_{\dot{\mathrm{NF}}\,0}(\mathrm{fmap}\ \dot{\mathrm{empty}}\ c)$$

$$\text{where}\ \dot{\mathrm{empty}} : \dot{\emptyset} \longrightarrow \dot{\mathrm{NF}}\ 0$$

$$\dot{\mathrm{empty}} = \mathrm{empty}$$

REIFICATION

$$\begin{array}{rl}
[\![0]\!]_\Delta = & \mathcal{C}\,\emptyset_\Delta \\
& \Big\downarrow \; \mathrm{fmap\ empty} \\
& \mathcal{C}(\dot{NF}\,0)_\Delta \\
& \Big\downarrow \; loc_{\dot{NF}} \\
NF\Delta\,0 = & \dot{NF}\,0_\Delta
\end{array}$$

Reification

$$reify_{A+B} : \mathcal{C}(\llbracket A \rrbracket \dot{\uplus} \llbracket B \rrbracket)_\Delta \longrightarrow NF\,\Delta\,(A+B)$$

$$reify_{A+B}\ c = loc_{\dot{NF}(A+B)}\left(fmap\ reifyBr\right)$$

$$\text{where } reifyBr : \llbracket A \rrbracket \dot{\uplus} \llbracket B \rrbracket \overset{\cdot}{\longrightarrow} \dot{NF}\,(A+B)$$

$$reifyBr\ (1, v_1) = inl\ (reify_A\ v_1)$$

$$reifyBr\ (2, v_2) = inl\ (reify_B\ v_2)$$

REIFICATION

$$
\begin{tikzcd}
 & {[\![A+B]\!]_\Delta = \mathcal{C}([\![A]\!] \dot{\uplus} [\![B]\!])_\Delta} \arrow[dl] \arrow[ddl, bend left, "\text{fmap reifyBr}"] \\
{\mathcal{C}(\dot{N}FA \dot{\uplus} NFB)_\Delta} \arrow[d] & \\
{\mathcal{C}(\dot{N}F(A+B))_\Delta} \arrow[d, "loc_{\dot{N}F(A+B)}"] & \\
{\dot{N}F(A+B)_\Delta} &
\end{tikzcd}
$$

$$\dot{N}F(A+B)_\Delta = NF\Delta(A+B)$$

INTERPRETING TERMS

$$[\![\text{abort } t]\!] \gamma = \text{loc}(\text{fmap empty}([\![t]\!] \gamma))$$

$$\begin{array}{c} [\![t]\!] \gamma : [\![0]\!]_\Delta = \mathcal{C}\emptyset_\Delta \\ \Big\downarrow {\scriptstyle \text{fmap empty}} \\ \mathcal{C}([\![A]\!])_\Delta \\ \Big\downarrow {\scriptstyle \text{loc}_{[\![A]\!]}} \\ [\![A]\!] \end{array}$$

Interpreting Terms

$$[\![\text{case } s\ (x_1 . t_1)\ (x_2 . t_2)]\!]\, \gamma = loc\,(fmap\ match\ v_{gs})$$

$$\text{where } \quad v_s = [\![s]\!]\, \gamma \ :\ \mathcal{C}([\![A]\!] \uplus [\![B]\!])_{\Delta}$$

$$v_{gs} = strength(\gamma, v_s) : \mathcal{C}([\![\Gamma]\!] \dot{\times} ([\![A]\!] \dot{\uplus} [\![B]\!]))_{\Delta}$$

$$match : [\![\Gamma]\!] \dot{\times} ([\![A]\!] \dot{\uplus} [\![B]\!]) \dot{\rightarrow} [\![C]\!]$$

$$match\,(\gamma', (1, v_1)) = [\![t_1]\!]\,(\gamma', v_1)$$

$$match\,(\gamma', (2, v_2)) = [\![t_2]\!]\,(\gamma', v_2)$$