INTRODUCTION TO TYPES & LAMBDAS

Nachi Valliappan (University of Edinburgh)

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Summary

$\mathsf{Ty}\ A, B ::= b \mid 1 \mid 0 \mid A + B \mid A \times B$ $\qquad$ $\mathsf{Ctx}\ \Gamma, \Delta ::= \cdot \mid \Gamma, x : A$

$\mathsf{Tm}\ \Gamma \vdash t : A$ $\qquad$ $\mathsf{Eq}\ \Gamma \vdash t = u : A$

LECTURE 1		LECTURE 2	
VAL	$\vdash_{\mathrm{VAL}} A$	NE	$\Gamma \vdash_{\mathrm{NE}} t : A$
		NF	$\Gamma \vdash_{\mathrm{NF}} t : A$
ENV	$\vdash_{\mathrm{ENV}} \Delta$	ENV	$\Gamma \vdash_{\mathrm{ENV}} \delta : \Delta$

WEAKENING

$$\boxed{\Gamma \sqsubseteq \Gamma'}$$

* $\cdot \sqsubseteq \cdot$

* $$\frac{\Gamma \sqsubseteq \Gamma'}{\Gamma \sqsubseteq \Gamma', x : A}\; x \notin \mathrm{dom}\,\Gamma'$$

* $$\frac{\Gamma \sqsubseteq \Gamma'}{\Gamma, x : A \sqsubseteq \Gamma', x : A}\; x \notin \mathrm{dom}\,\Gamma'$$

Summary

Lecture 1

$[\![A]\!], [\![\Gamma]\!] \in Set$

$[\![t]\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!]$

$reflect_A : Val\ A \rightarrow [\![A]\!]$

$reflect_\Gamma : Env\ \Gamma \rightarrow [\![\Gamma]\!]$

$reify_A : [\![A]\!] \rightarrow Val\ A$

$eval\ t : Env\ \Gamma \rightarrow Val\ A$

Lecture 2

$[\![A]\!], [\![\Gamma]\!] : Ctx \rightarrow Set$

$[\![t]\!] : [\![\Gamma]\!] \overset{\cdot}{\longrightarrow} [\![A]\!]$

$reflect_A : NE\ A \overset{\cdot}{\rightarrow} [\![A]\!]$

$reflect_\Gamma : Env\ \Gamma \overset{\cdot}{\rightarrow} [\![\Gamma]\!]$

$reify_A : [\![A]\!] \overset{\cdot}{\rightarrow} NF\ A$

$eval\ t : Env\ \Gamma \overset{\cdot}{\rightarrow} NF\ A$

DOES NORMALISATION MATTER?

APPLICATIONS

* Deciding equality:

$$t = u \iff [\![t]\!] = [\![u]\!]$$

$$\begin{tikzcd}[column sep=small] t \arrow[dr] & & u \arrow[dl] \\ & ? & \end{tikzcd}$$

* Refuting existence of terms

* Program synthesis & proof search

EXAMPLES IN ACTION

UPWARD CLOSURE ("PRESHEAF")

Def: A family of sets X is upward closed when for all $\Gamma \sqsubseteq \Gamma'$, it is equipped with $\uparrow_X : X_\Gamma \longrightarrow X_{\Gamma'}$

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LOCALISING ("$\mathcal{C}$-MONAD ALGEBRAS")

Def: A family of sets X is localising when it is equipped with $loc_X : \mathcal{C} X \dot{\to} X$

Soundness

Showing $t = u$ implies $[\![t]\!] = [\![u]\!]$

Observation :

$[\![A]\!]$ and $[\![\Gamma]\!]$ are upward closed and localising families

$[\![t]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$ are families of functions

Intention :

$[\![A]\!]$ and $[\![\Gamma]\!]$ are presheaves and ℓ-monad algebras

$[\![t]\!] : [\![\Gamma]\!] \dot{\to} [\![A]\!]$ are natural transformations

Functions

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \mathrm{lam}\, x.\, t : A \to B} \qquad \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathrm{app}\; t\; u : B}$$

$(\beta_{\to})$ $\mathrm{app}\,(\mathrm{lam}\, x.\, t)\; u = t[u/x]$

$(\eta_{\to})$ For $\Gamma \vdash t : A \to B$, $t = \mathrm{lam}\, x.\ \mathrm{app}\,(\uparrow t)\; x$
(and $x \notin \mathrm{dom}\, \Gamma$)

Functions

$$\frac{\Gamma, x : A \vdash_{NF} t : B}{\Gamma \vdash_{NF} \mathrm{lam}\, x.\, t : A \to B} \qquad \frac{\Gamma \vdash_{NE} t : A \to B \quad \Gamma \vdash_{NF} u : A}{\Gamma \vdash_{NE} \mathrm{app}\; t\; u : B}$$

FUNCTIONS

$$\frac{\Gamma, x : A \vdash_{NF} t : B}{\Gamma \vdash_{NF} \mathrm{lam}\ x.\ t : A \to B} \qquad \frac{\Gamma \vdash_{NE} t : A \to B \quad \Gamma \vdash_{NF} u : A}{\Gamma \vdash_{NE} \mathrm{app}\ t\ u : B}$$

$$[\![A \to B]\!]_{\Delta} = \{ [\![A]\!]_{\Delta'} \longrightarrow [\![B]\!]_{\Delta'} \mid \forall \Delta' \sqsupseteq \Delta \}$$

FUNCTIONS

$$\frac{\Gamma, x : A \vdash_{NF} t : B}{\Gamma \vdash_{NF} \mathrm{lam}\ x.\, t : A \to B} \qquad \frac{\Gamma \vdash_{NE} t : A \to B \quad \Gamma \vdash_{NF} u : A}{\Gamma \vdash_{NE} \mathrm{app}\ t\ u : B}$$

$$[\![A \to B]\!]_{\Delta} = \left\{ [\![A]\!]_{\Delta'} \longrightarrow [\![B]\!]_{\Delta'} \;\middle|\; \forall \Delta' \sqsupseteq \Delta \right\}$$

$$\mathrm{reflect}_{A \to B}\ t = \lambda a.\ \mathrm{reflect}_{B}\ (\mathrm{app}\ (\uparrow t)\ (\mathrm{reify}_{B}\ a))$$

$$\mathrm{reify}_{A \to B}\ f = \mathrm{lam}\ x.\ \mathrm{reify}_{B}\ (f\ (\mathrm{reflect}_{A}\ x))$$

MOGGI'S MONADIC METALANGUAGE

$\mathrm{Ty}\ A, B ::= \ldots \mid TA$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathrm{return}\ t : TA}$$

$$\frac{\Gamma \vdash t : TA \qquad \Gamma, x : A \vdash u : TB}{\Gamma \vdash \mathrm{let}\ x := t\ \mathrm{in}\ u : TB}$$

Moggi's Monadic Metalanguage

(β_{TA}) $\text{let } x := \text{return } t \text{ in } u \;=\; u[t/x]$

(η_{TA}) For $\Gamma \vdash t : TA$,

$$t = \text{let } x := t \text{ in } (\text{return } x)$$

(α_{TA}) $\text{let } x := (\text{let } y := t_1 \text{ in } t_2) \text{ in } t_3$

$$= \text{let } y := t_1 \text{ in } (\text{let } x = \uparrow t_2 \text{ in } t_3)$$

MOGGI'S MONADIC METALANGUAGE

$$\frac{\Gamma \vdash_{NF} t : A}{\Gamma \vdash_{NF} \mathrm{return}\ t : TA}$$

$$\frac{\Gamma \vdash_{NE} t : TA \qquad \Gamma, x : A \vdash_{NF} u : TB}{\Gamma \vdash_{NF} \mathrm{let}\ x := t\ \mathrm{in}\ u : TB}$$

EXTENDING THE MONADIC METALANGUAGE

$$\frac{\Gamma \vdash t_1 : TA \quad \ldots \quad \Gamma \vdash t_n : TA}{\Gamma \vdash op(t_1, \ldots, t_n) : TA}$$

$$(\mathcal{E}_{TA}) \quad \text{let } x := op(t_1, \ldots, t_n) \text{ in } u$$

$$= op(\text{let } x := t_1 \text{ in } u, \ldots, \text{let } x = t_n \text{ in } u)$$

EXERCISE

* Define the terms :

$$\Gamma \vdash fmap : (A \to B) \to TA \to TB$$

$$\Gamma \vdash point : A \to TA$$

$$\Gamma \vdash join : TTA \to TA$$

EXERCISE

Given $\Gamma \vdash f : A \to B$, show the diagrams commute:

$$
\begin{tikzcd}
A \arrow[r, "\mathrm{point}"] \arrow[d, "f"'] & TA \arrow[d, "\mathrm{fmap}\, f"] \\
B \arrow[r, "\mathrm{point}"'] & TB
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2A \arrow[r, "\mathrm{join}"] \arrow[d, "\mathrm{fmap}^2 f"'] & TA \arrow[d, "\mathrm{fmap}\, f"] \\
T^2B \arrow[r, "\mathrm{join}"'] & TB
\end{tikzcd}
$$

$(\mathrm{point} \circ f = \mathrm{fmap}\, f \circ \mathrm{point})$ $(\mathrm{join} \circ \mathrm{fmap}^2 f = \mathrm{fmap}\, f \circ \mathrm{join})$

Fitch-style modal calculus

$$\mathsf{Ty}\quad A, B ::= \ldots \mid \Box A \qquad \mathsf{Ctx}\quad \Gamma, \Delta ::= \ldots \mid \Gamma, \text{🔒}$$

$$\frac{\Gamma, \text{🔒} \vdash t : A}{\Gamma \vdash \mathsf{box}\ t : \Box A} \qquad \frac{\Gamma \vdash t : \Box A}{\Gamma' \vdash \mathsf{unbox}\ t : A}\ \Gamma, \text{🔒} \sqsubseteq \Gamma'$$

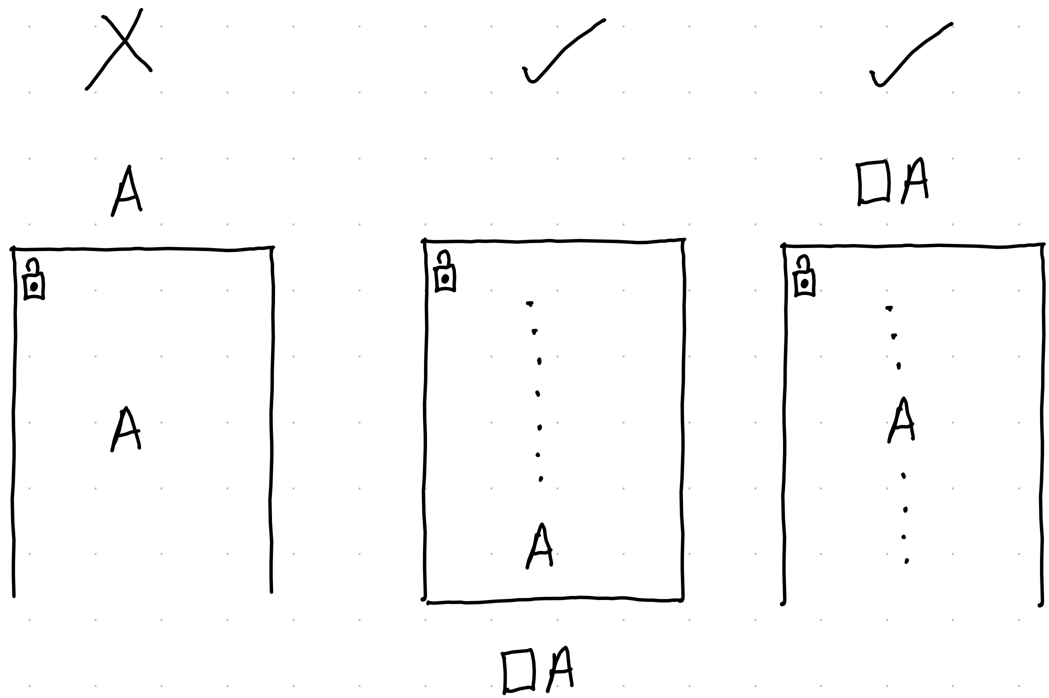
$$\boxed{\Gamma \sqsubseteq \Gamma'} \qquad \ldots \qquad \frac{\Gamma \sqsubseteq \Gamma'}{\Gamma, \text{🔒} \sqsubseteq \Gamma', \text{🔒}}$$

TYPICALLY

$$\frac{}{\Gamma, x : A, \Gamma' \vdash_v x : A}\ \text{🔒} \notin \Gamma'$$

BUT WE DON'T NEED THAT!

Strict subordinate proofs



Fitch-style modal calculus

$$(\beta_{\Box A}) \quad \mathrm{unbox}\,(\mathrm{box}\ t) = {\uparrow} t$$

$$(\eta_{\Box A}) \quad \text{For } \Gamma \vdash t : \Box A, \quad t = \mathrm{box}\,(\mathrm{unbox}\ t)$$

$$\frac{\Gamma, \text{🔒} \vdash_{NF} t : A}{\Gamma \vdash_{NF} \mathrm{box}\ t : \Box A} \qquad \frac{\Gamma \vdash_{NE} t : \Box A}{\Gamma' \vdash_{NE} \mathrm{unbox}\ t : A}\ \Gamma, \text{🔒} \sqsubseteq \Gamma'$$

EXERCISE

* Show the necessitation rule is admissible

$$\frac{\cdot \vdash t : A}{\Gamma \vdash \mathrm{nec}\ t : \Box A}$$

* Show that axiom K is admissible

$$\Gamma \vdash \mathrm{axK} : \Box(A \to B) \to \Box A \to \Box B$$

* Why not extend the program calculus directly with constructs 'nec' and 'axK'?

THINGS I WISH WE HAD TIME FOR

* Denotational semantics and normalisation for the calculi with monads and boxes

* Normalisation without η or π rules

* Categorical semantics and term models

THINGS I WISH WE HAD TIME FOR

* "Pattern matching" eliminators

$$\frac{\Gamma \vdash t : TA \qquad \Gamma, x : A \vdash u : TB}{\Gamma \vdash \mathrm{let}\ x := t\ \mathrm{in}\ u : TB}$$

$$\frac{\Gamma \vdash s : A \vee B \qquad \Gamma, x_1 : A \vdash t_1 : C \qquad \Gamma, x_2 : A \vdash t_2 : C}{\Gamma \vdash \mathrm{case}\ s\ (x_1 . t)\ (x_2 . t_2) : C}$$

THINGS WE HAD TIME FOR

THINGS WE HAD TIME FOR

$$\begin{tikzcd}
 & D, 1, 0, \times, + \arrow[dl] \arrow[d] \arrow[dr] & \\
\to & T & \square
\end{tikzcd}$$

Detours, conversions, Normal forms, subformula property, type-driven design.

Semantics
- Sets
- Families of sets

Interpreters
- closed
- open

THINGS WE HAD TIME FOR

- ALGEBRA AND NORMALISATION
- CATEGORY THEORY
- MODAL LOGICS

representation theorems

algebraic semantics

$$\left\{ \begin{array}{c} [\![A]\!] \subseteq W \text{ or } [\![A]\!] : W \to 2 \\ \text{vs} \\ [\![A]\!] : \underset{\underset{Ctx}{\shortparallel}}{W} \to Set \end{array} \right\}$$

completeness

vs

normalisation

THANK YOU SPLV '26