Lax modal lambda calculi

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Classical modal logic

Classical propositional logic +

$$\frac{\cdot \vdash A}{\Gamma \vdash \Box A} \text{ Necessitation}$$

$$\Box(A \to B) \to \Box A \to \Box B \text{ Axiom K}$$

$$\Diamond A \equiv \neg \Box \neg A$$

Intuitionistic modal logic (IML)

$\Box$ and $\Diamond$ are independent in IML

...as are $\wedge$ and $\vee$ in IPL

The most basic IML: $CK_\Box$

Intuitionistic propositional logic +

$$\frac{\cdot \vdash A}{\Gamma \vdash \Box A} \textsc{Necessitation}$$

$$\Box(A \to B) \to \Box A \to \Box B \ \textsc{Axiom K}$$

IMLs with boxes [Božić & Došen 1984,...]

Intuitionistic propositional logic + Nec. +

$\mathrm{CS4}_{\Box}$ $\begin{cases} \mathrm{K}\colon \Box(A \to B) \to \Box A \to \Box B \\ \mathrm{T}\colon \Box A \to A \\ 4\colon \Box A \to \Box\Box A \end{cases}$

$\mathrm{K} \longrightarrow \mathrm{CK}_{\Box}$

$\mathrm{K} + \mathrm{T} \longrightarrow \mathrm{CT}_{\Box}$

$\mathrm{K} + 4 \longrightarrow \mathrm{CK4}_{\Box}$

Lambda calculi with boxes [Borghuis 1994, Clouston 2018]

Simply-typed λ-calculus $+$

$\lambda_{\mathrm{CS4}_\Box}$ $\left\{\begin{array}{l} \mathrm{K}\colon \Box(A \to B) \to \Box A \to \Box B \\ \mathrm{T}\colon \Box A \to A \\ 4\colon \Box A \to \Box\Box A \end{array}\right.$

$\mathrm{K}\colon \Box(A \to B) \to \Box A \to \Box B \longrightarrow \lambda_{\mathrm{CK}_\Box}$

$\mathrm{K} + \mathrm{T}\colon \Box A \to A \longrightarrow \lambda_{\mathrm{CT}_\Box}$

$\mathrm{K} + 4\colon \Box A \to \Box\Box A \longrightarrow \lambda_{\mathrm{CK4}_\Box}$

I will not fall for the quasi-philosophical trap

IML with diamonds: Lax logic

Intuitionistic propositional logic +

$$\text{LL}\begin{cases} \text{S: } A \times \Diamond B \to \Diamond(A \times B) \\ \text{R: } A \to \Diamond A \\ \text{J: } \Diamond\Diamond A \to \Diamond A \end{cases}$$

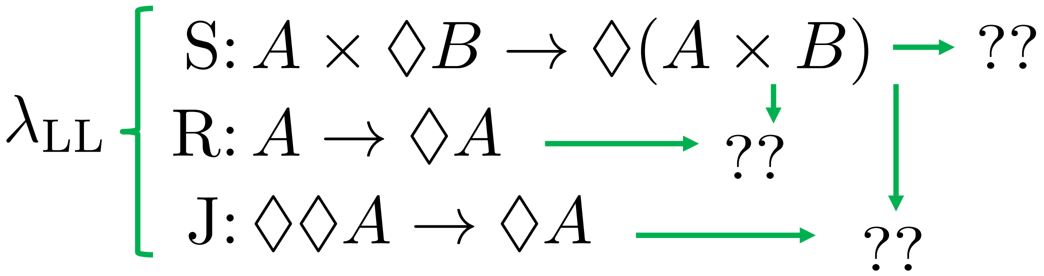
$$\frac{\cdot \vdash A \to B}{\Gamma \vdash \Diamond A \to \Diamond B}$$

Moggi's monadic metalanguage

Simply-typed λ-calculus +

$$\lambda_{\mathrm{LL}} \left\{ \begin{array}{l} \mathrm{S}\colon A \times \Diamond B \to \Diamond(A \times B) \\ \mathrm{R}\colon A \to \Diamond A \\ \mathrm{J}\colon \Diamond\Diamond A \to \Diamond A \end{array} \right.$$

An objective of this talk



Base calculus is STLC, nothing funky

$$\text{Ty } A, B ::= \tau \mid A \to B \mid A \times B \mid \Diamond A$$

$$\text{Ctx } \Gamma ::= \cdot \mid \Gamma, x : A$$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.\, t : A \to B}$$

$$\frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\, u : B}$$

Calculus for SL

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, x : A \vdash u : B}{\Gamma \vdash \ \mathbf{letmap}\ x = t\ \mathbf{in}\ u : \Diamond B}$$

$$\vdash \lambda x.\, \mathbf{letmap}\ y = \mathbf{snd}\ x\ \mathbf{in}\ (\mathbf{pair}\ (\mathbf{fst}\ x)\ y) : A \times \Diamond B \to \Diamond(A \times B)$$

Calculus for SRL

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, x : A \vdash u : B}{\Gamma \vdash \ \textbf{letmap}\ x = t\ \textbf{in}\ u : \Diamond B}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \textbf{return}\ t : \Diamond A}$$

$$\cdot \vdash \lambda x.\,\textbf{return}\ x : A \to \Diamond A$$

Calculus for SJL

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, x : A \vdash u : B}{\Gamma \vdash \ \textbf{letmap}\ x = t\ \textbf{in}\ u : \Diamond B}$$

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, x : A \vdash u : \Diamond B}{\Gamma \vdash \ \textbf{let}\ x = t\ \textbf{in}\ u : \Diamond B}$$

$$\cdot \vdash \lambda x.\, \textbf{let}\ y = x\ \textbf{in}\ y : \Diamond\Diamond A \to \Diamond A$$

Calculus for LL

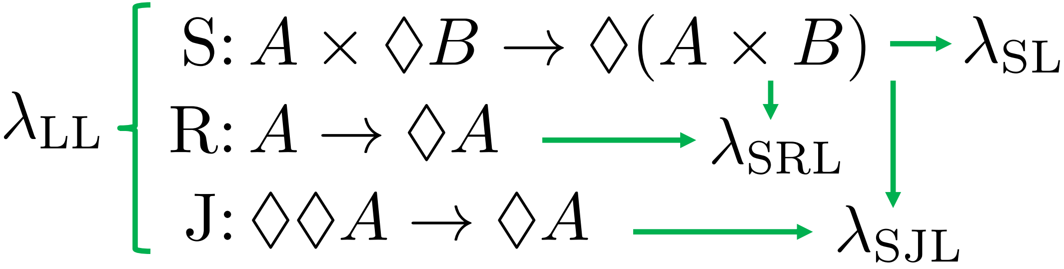
$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \textbf{return}\ t : \Diamond A}$$

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, x : A \vdash u : \Diamond B}{\Gamma \vdash \ \textbf{let}\ x = t\ \textbf{in}\ u : \Diamond B}$$

$$\cdot \vdash \lambda x.\,\textbf{let}\ y = \textbf{snd}\ x\ \textbf{in}\ \textbf{return}\ (\textbf{pair}\ (\textbf{fst}\ x)\ y) : A \times \Diamond B \to \Diamond(A \times B)$$

What have I really said?

Lax modal lambda calculi



How do lax lambda calculi correspond to lax logics?

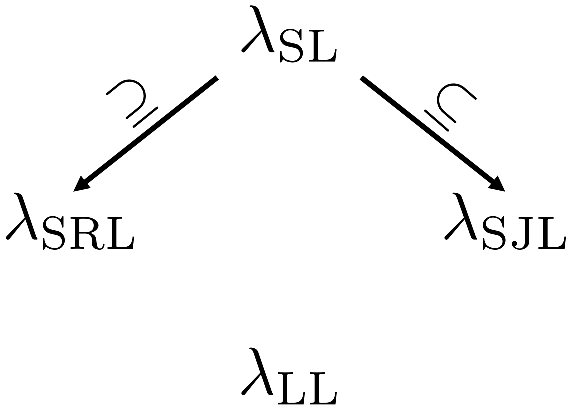
What does correspondence mean?

Meta-theoretic hygiene

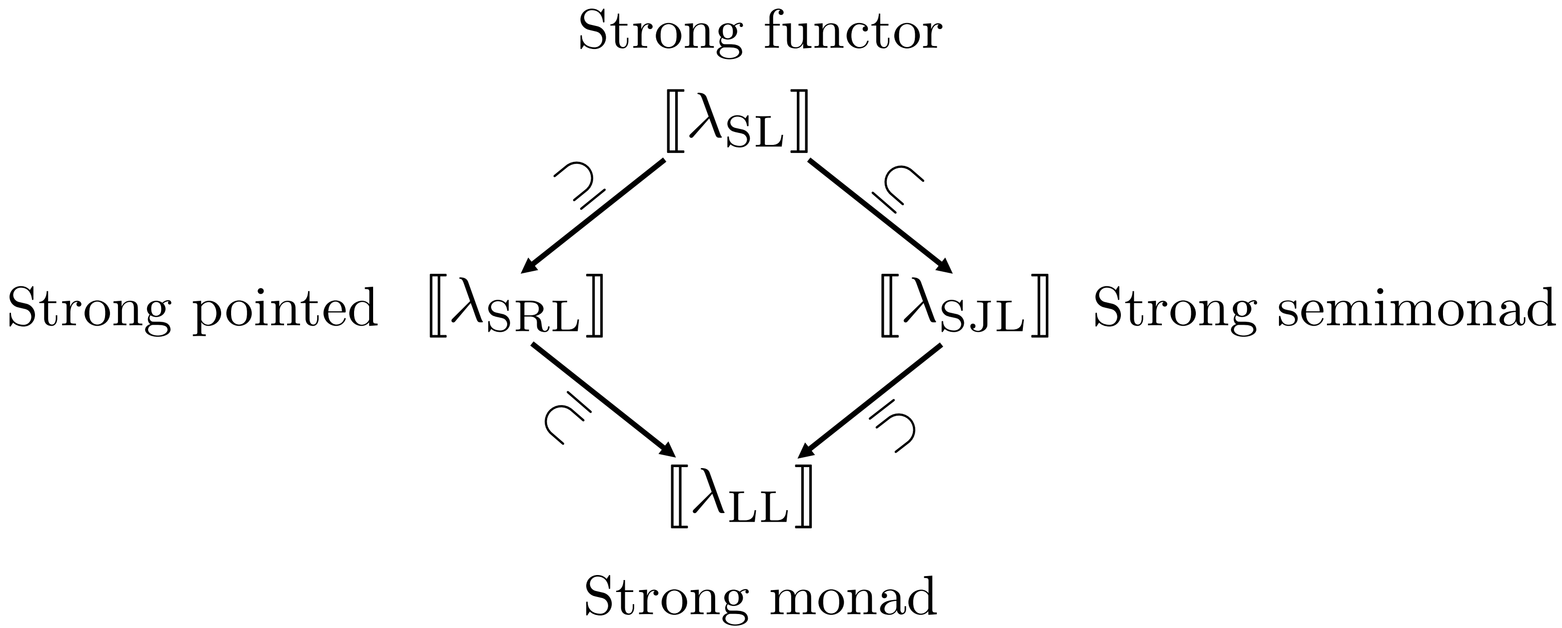
- Do the calculi accidentally admit garbage axioms?
- Can we extract proofs from terms?
- Are the calculi normalizing?
- Are the equational theories decidable?

Answering these for each calculus is daunting

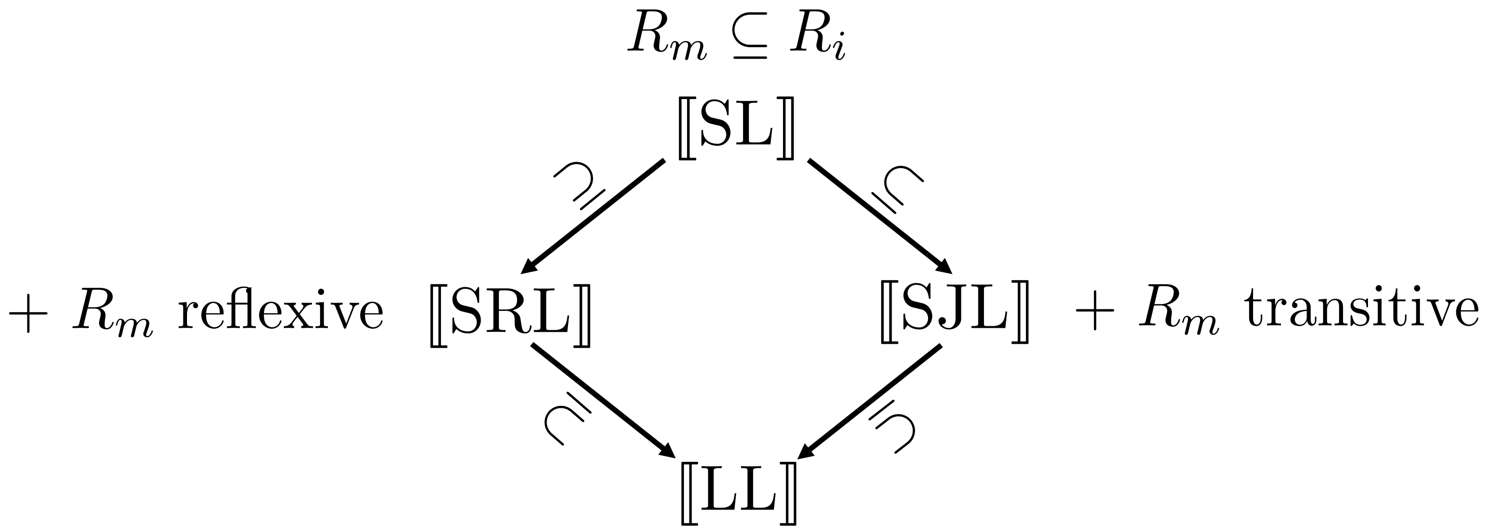
Syntax is not incremental



Semantics is!



Semantics is!



How do the different semantics correspond?

Possible-world semantics

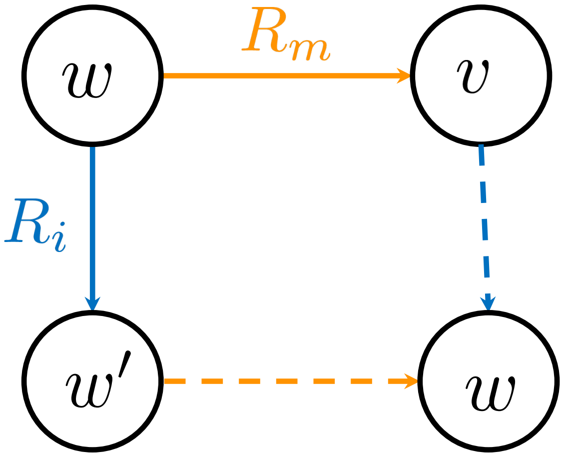
Frame: $(\mathcal{W}, R_i, R_m)$

$R_i^{-1}; R_m \subseteq R_m; R_i^{-1}$

$R_m \subseteq R_i$

R_m reflexive

R_m transitive



$$[\![A \to B]\!]_w = \forall w'.\, w \; R_i \; w' \to [\![A]\!]_{w'} \to [\![B]\!]_{w'}$$

$$[\![\Diamond A]\!]_w = \exists v.\, w \; R_m \; v \times [\![A]\!]_v$$

Possible-world semantics

$$R_i^{-1}; R_m \subseteq R_m; R_i^{-1} \implies w\ R_i\ w' \to [\![\Diamond A]\!]_w \to [\![\Diamond A]\!]_{w'}$$

$$R_m \subseteq R_i \implies [\![A \times \Diamond B \to \Diamond(A \times B)]\!]_w$$

$$R_m \text{ reflexive} \implies [\![A \to \Diamond A]\!]_w$$

$$R_m \text{ transitive} \implies [\![\Diamond\Diamond A \to \Diamond A]\!]_w$$

Proof-relevant possible-world semantics

$R_i^{-1}; R_m \subseteq R_m; R_i^{-1}$	$\Longrightarrow$	$\Diamond$ is a presheaf functor
$R_m \subseteq R_i$	$\Longrightarrow$	$\Diamond$ is strong
R_m reflexive	$\Longrightarrow$	$\Diamond$ is pointed
R_m transitive	$\Longrightarrow$	$\Diamond$ is a semimonad

* conditions apply

The Trick

Frame $(\mathcal{W}, R_i, R_m)$ $\Longrightarrow$ Presheaf category $\widehat{\mathcal{W}}$

$\searrow$ $\swarrow$

Cartesian-closed category $\widehat{\mathcal{W}}$

It’s an old trick

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Kripke-style models for typed lambda calculus

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Communicated by A. Nerode

It's an old trick [Mitchell and Moggi 1991]

intuitionistic completeness theorem (see, e.g., [4, 13, 14]). However, we prefer the completeness theorem using only Kripke models for several reasons. For one, Kripke models are relatively easy to picture, and they seem to support a set-like intuition about the lambda terms better than arbitrary cartesian closed categories. In addition, predicate logic may be interpreted over Kripke lambda models, while there is no analogous interpretation in arbitrary cartesian closed categories (except indirectly via the Yoneda embedding). A practical advantage is that it is often easy to devise Kripke counter-models to implications like (*). Finally, the useful techniques of logical relations generalize to Kripke lambda models without much difficulty and provide an easy way to construct Kripke lambda models from Henkin-like structures.

Beyond boxes and diamonds, using neighborhoods

Frame $(\mathcal{W}, R_i, N)$ $N : W \to \mathcal{P}(\mathcal{P}(W))$

$$\mathcal{C}P_w = \Sigma_n .\, n \in N(w) \times \forall v .\, v \in n \to P_v$$

EOM