Lax Modal Lambda Calculi

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Abstract

Intuitionistic modal logic (IML) is the study of extending intuitionistic propositional logic with the box and diamond modalities. Advances in IML have led to a plethora of useful applications in programming languages via the development of corresponding type theories with modalities. Until recently, IMLs with diamonds have been misunderstood as somewhat peculiar and unstable, causing the development of type theories with diamonds to lag behind type theories with boxes. In this article, we develop a family of typed-lambda calculi corresponding to sublogics of a peculiar IML with diamonds known as Lax logic. These calculi provide a modal logical foundation for various strong functors in typed-functional programming. We present possible-world and categorical semantics for these calculi and constructively prove normalization, equational completeness and proof-theoretic inadmissibility results. Our key results have been formalized using the proof assistant Agda.

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1 Introduction

In modal logic, a modality is a unary logical connective that exhibits some logical properties. Two such modalities are the connectives $\Box$ ("box") and $\Diamond$ ("diamond"). Intuitively, a formula $\Box A$ can be understood as "necessarily A" and a formula $\Diamond A$ as "possibly A". In classical modal logic, the most basic logic K extends classical propositional logic (CPL) with the box modality, the *necessitation* rule (if A is a theorem then so is $\Box A$) and the K axiom $(\Box(A \Rightarrow B) \Rightarrow \Box A \Rightarrow \Box B)$. The diamond modality can be encoded in this logic as a dual of the box modality: $\Diamond A \equiv \neg\Box\neg A$. That is, $\Diamond A$ is true if and only if $\neg\Box\neg A$ is true.

In intuitionistic modal logic (IML), there is no consensus on one logic as the most basic logic. We instead find a variety of different IMLs based on different motivations. The $\Box$ and $\Diamond$ modalities are independent connectives in IML [35, Requirement 5], just as $\wedge$ and $\vee$ are independent connectives that are not inter-definable in intuitionistic propositional logic (IPL). In contrast to $\Box$, however, the logical properties of $\Diamond$ vary widely in IML literature. This has misconstrued $\Diamond$ as a controversial and unstable modality. It had been incorrectly assumed until recently that several IMLs with both $\Box$ and $\Diamond$ coincided (i.e. were conservative extensions of their sublogics) only in the $\Diamond$-free fragment, suggesting some sort of stability of $\Box$-only logics. Fortunately, misconceptions around intuitionistic diamonds have been broken in recent results [16, 18] and we are approaching a better understanding of it.

Advances in IML have led to a plethora of useful applications in programming languages through the development of corresponding type theories with modalities. Modal lambda calculi [32, 13] with box modalities have found applications in staged meta-programming [17, 31, 23], reactive programming [5], safe usage of temporal resources [2] and checking productivity of recursive definitions [10]. Two particular box axioms that have received plenty of attention in these developments are the axioms $\mathrm{T} : \Box A \Rightarrow A$ and $4 : \Box A \Rightarrow \Box\Box A$. Dual-context modal calculi [32, 24] which admit one or both of these axioms are well-understood. These calculi enjoy a rich meta-theory, including confluent reduction, normalization and a comprehensive analysis of provability. Fitch-style modal lambda calculi [13] admitting axioms T and 4 further enjoy an elegant categorical interpretation, possible-world

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semantics, and results showing how categorical models of these calculi can be constructed using possible-world semantics of their corresponding logics [37].

Lambda calculi with diamond modalities in comparison have received much less attention from the type-theoretic perspective. The controversy surrounding the diamond modality in IML appears to have restricted the development of type theories with diamonds. For example, Kavvos [25] cites Simpson's survey [35] of IMLs and restricts the development of dual-context modal calculi "to the better-behaved, and seemingly more applicable box modality" arguing that the "computational interpretation [of $\Diamond$] is not very crisp". Recent breakthroughs in intuitionistic modal logic have made it clear that diamonds are no more problematic than boxes. In this article, we further the type-theoretic account of a special class of diamond modalities with compelling applications in programming languages.

Propositional lax logic, or simply lax logic (LL), is an intuitionistic modal logic introduced independently by Fairtlough and Mendler [20] and Benton, Bierman and de Paiva [7]. LL extends IPL with a diamond modality $\Diamond$, known as the lax modality, which exhibits a peculiar modal axiom S (for "strength"), in addition to axioms R (for "return") and J (for "join") that are well-known as classical duals to the box axioms T and 4 respectively.

$$\mathrm{S} : A \times \Diamond B \Rightarrow \Diamond(A \times B) \qquad \mathrm{R} : A \Rightarrow \Diamond A \qquad \mathrm{J} : \Diamond\Diamond A \Rightarrow \Diamond A$$

It is known that LL corresponds to a typed-lambda calculus (we call λ_{LL}) known as Moggi's *monadic metalanguage* [30], which models side effects in functional programming using *strong monads* from category theory. Benton, Bierman and de Paiva [7], and later Pfenning and Davies [32], show that a judgement is provable in a natural deduction proof system for LL if and only if there exists a typing derivation for its corresponding judgement in λ_{LL}. However, in contrast to the comprehensive treatment of box modalities mentioned above, there remain several gaps in our understanding of the lax modality:

1. It has remained unclear as to whether type theories can exist for sublogics of LL or whether the axioms of LL in combination happen to coincidentally enjoy a status of "well-behavedness". What happens if we drop one or more of the modal axioms R and J? Does a corresponding type theory still exist?
2. A satisfactory account of the correspondence between the possible-world semantics of LL and the categorical semantics of λ_{LL} is still missing. In particular, how can we leverage the possible-world semantics of LL to construct models of λ_{LL}?

The first objective of this article is to develop corresponding type theories for sublogics of LL that drop one or both of axioms R and J. From the type-theoretic perspective, this corresponds to type theories for non-monadic strong functors, which are prevalent in functional programming. For example, in Haskell, the array data type (in `Data.Array`) is a strong functor that neither exhibits return (axiom R) nor join (axiom J). Several other Haskell data types exhibit return[1] or join[2], but not both[3]. We are interested in developing a uniform modal logical foundation for the axioms of non-monadic strong functors.

The second objective of this article is to study the connection between possible-world semantics of LL and its sublogics and categorical models of their corresponding type theories. Possible-world semantics for logics are concerned with provability of formulas and not about proofs themselves. Categorical models of lambda calculi, on the other hand, distinguish

[1] https://hackage.haskell.org/package/pointed-5.0.5/docs/Data-Pointed.html
[2] https://hackage.haskell.org/package/semigroupoids-6.0.1/docs/Data-Functor-Bind.html#g:4
[3] https://wiki.haskell.org/Why_not_Pointed%3F

different proofs (terms) of the same proposition (type). Mitchell and Moggi [29] show the connection between these two different semantics using a categorical refinement of possible-world semantics for the simply-typed lambda calculus (STLC). They note that their refined semantics, which we shall call *proof-relevant possible-world semantics*, makes it "easy to devise Kripke counter-models" since they "seem to support a set-like intuition about lambda terms better than arbitrary cartesian closed categories". We wish to achieve this technical convenience in model construction for all the modal lambda calculi in this article.

In this article, towards our first objective, we formulate three new modal lambda calculi as subsystems of λ_{LL}: λ_{SL}, λ_{SRL}, λ_{SJL}. The calculus λ_{SL} models *strong* functors and corresponds to a logic SL (for "S-lax Logic") that admits axiom S, but neither R nor J. The calculus λ_{SRL} models strong *pointed* functors and corresponds to a logic SRL (for "SR-lax Logic") that admits axioms S and R, but not J. The calculus λ_{SJL} models strong *semimonads* and corresponds to a logic SJL (for "SJ-lax Logic") that admits axioms S and J, but not R. We refer to all four calculi collectively as *lax modal lambda calculi*. Towards our second objective, we extend Mitchell and Moggi's proof-relevant possible-world semantics to lax modal lambda calculi and show that it is complete for their equational theories. We further show that all four calculi are normalizing by constructing *Normalization by Evaluation* models as instances of possible-world semantics and prove completeness and inadmissibility results as corollaries. All the theorems in this paper have been verified using the proof assistant Agda [1] and the formalization can be found at: `https://anonymous.4open.science/r/s-C71D/README.md`.

2 Overview of LL and its corresponding lambda calculus λ_{LL}

In this section, we will define the syntax and semantics of LL and its sublogics that extend the so-called *negative*, i.e. disjunction and absurdity-free, fragment of IPL. This section is a recap of known results from previously published work alongside a discussion of technical background presumed in the rest of this article.

2.1 Syntax and semantics of LL

Syntax. The language of (the negative fragment of) LL consists of formulas defined inductively by propositional atoms (p, q, r, etc.), a constant $\top$ and logical connectives $\times$, $\Rightarrow$ and $\Diamond$.

$$\textsf{Prop} \quad A, B := p, q, r, \ldots \mid \top \mid A \times B \mid A \Rightarrow B \mid \Diamond A \qquad \textsf{Ctx} \quad \Gamma, \Delta := \cdot \mid \Gamma, A$$

The constant $\top$ denotes universal truth, the binary connectives $\times$ and $\Rightarrow$ respectively denote conjunction and implication, and the unary connective $\Diamond$ denotes the lax modality. Intuitively, a formula $\Diamond A$ may be understood as qualifying the truth of formula A under *some* constraint. A context Γ is a multiset of formulas $A_1, A_2, ..., A_n$, where $\cdot$ denotes the empty context.

A Hilbert-style axiomatisation of LL can be given by extending the usual axioms and rules of deduction for IPL with the modal axioms S, R, and J in Section 1.

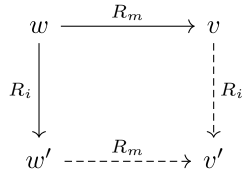
Semantics. The possible-world semantics of LL defines the truth of LL-formulas in a model using gadgets known as *frames*. An LL-frame $F = (W, R_i, R_m)$ is a triple that consists of a set W of *worlds* and two reflexive transitive relations R_i (for "intuitionistic") and R_m (for "modal") on worlds satisfying two compatibility conditions:

- Forward confluence: ${R_i}^{-1}; R_m \subseteq R_m; {R_i}^{-1}$
- Inclusion: $R_m \subseteq R_i$

The relation ${R_i}^{-1}$ is the converse of R_i and ; denotes composition of relations. The composite of two binary relations R_1 and R_2 on worlds is defined as $R_1; R_2 = \{(x, z) \mid \text{there exists } y \in W \text{ such that } (x, y) \in R_1 \text{ and } (y, z) \in R_2\}$.

We may intuitively understand worlds as nodes in a graph denoting the "state of assumptions", relation R_i as paths denoting increase in assumptions, and relation R_m as paths denoting constraining of assumptions. That is, $w\ R_i\ w'$ denotes the increase in assumptions from world w to w', and $w\ R_m\ v$ denotes a constraining of w by v such that v is reachable from w when the constraint can be satisfied. The inclusion condition $R_m \subseteq R_i$ means imposing a constraint increases assumptions.

The forward confluence condition ${R_i}^{-1}; R_m \subseteq R_m; {R_i}^{-1}$ states that constraints can be "transported" over an increase in assumptions. It can be visualized as depicted on the right, where the dotted lines represent "there exists". This condition does not appear in Fairtlough and Mendler's original work [20], but can be found in earlier work on intuitionistic diamonds by Božić and Došen [12, §8] and Plotkin and Stirling [33]. It simplifies the interpretation of $\Diamond$ and is satisfied by all the models we will construct in this article to prove completeness. We return to the discussion on forward confluence in Section 6.



A model $\mathcal{M} = (F, V)$ couples a frame F with a *valuation* function V that assigns to each propositional atom p a set $V(p)$ of worlds hereditary in R_i, i.e. if $w\ R_i\ w'$ and $w \in V(p)$ then $w' \in V(p)$. The truth of a formula in a model M is defined by the *satisfaction* relation $\Vdash$ for a given world $w \in W$ by induction on a formula as:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & w \in V(p) \\
\mathcal{M}, w \Vdash \top & \text{iff} & \text{true} \\
\mathcal{M}, w \Vdash A \times B & \text{iff} & \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B \\
\mathcal{M}, w \Vdash A \Rightarrow B & \text{iff} & \text{for all } w' \in W \text{ such that } w\ R_i\ w',\ \mathcal{M}, w' \Vdash A \text{ implies } \mathcal{M}, w' \Vdash B \\
\mathcal{M}, w \Vdash \Diamond A & \text{iff} & \text{there exists } v \in W \text{ with } w\ R_m\ v \text{ and } \mathcal{M}, v \Vdash A
\end{array}
$$

We write $\mathcal{M} \models A$ to mean $\mathcal{M}, w \Vdash A$ at all worlds w, and $\mathcal{M} \models \Gamma$ to mean that $\mathcal{M} \models A_i$ for all formulas A_i with $1 \leq i \leq n$ in context $\Gamma = A_1, ...A_n$. Furthermore, we write $\Gamma \models A$ to mean $\mathcal{M} \models \Gamma$ implies $\mathcal{M} \models A$ for all models $\mathcal{M}$.

The soundness of the semantics of LL can be shown using the following key properties:

▶ **Proposition 1.** *Given an arbitrary model $\mathcal{M} = (F, V)$ of LL*
- *if $w\ R_i\ w'$ and $\mathcal{M}, w \Vdash A$ then $\mathcal{M}, w' \Vdash A$, for all worlds w, w' and formulas A*
- *$\mathcal{M} \models A \times \Diamond B \Rightarrow \Diamond(A \times B)$ for all worlds w and formulas A, B*
- *$\mathcal{M} \models A \Rightarrow \Diamond A$ for all worlds w and formulas A*
- *$\mathcal{M} \models \Diamond\Diamond A \Rightarrow \Diamond A$ for all worlds w and formulas A*

Proof. The first property, known as the monotonicity lemma states that the truth of an arbitrary formula is retained as knowledge increases. This lemma is proved as usual by induction on formulas, using the forward confluence condition for the case of $\Diamond A$. The remaining properties are shown using the definition of the satisfaction clause by respectively using the inclusion condition $R_m \subseteq R_i$, reflexivity of R_m, and transitivity of R_m ◀

2.2 Syntax and semantics of λ_{LL}

Syntax. The monadic meta-language, or λ_{LL}, was developed by Moggi [30] independently before LL. The calculus λ_{LL} can be presented as an extension of STLC featuring cartesian products with a unary type constructor $\Diamond$ that exhibits axioms S, R and J.

$$
\textit{Ty} \quad A, B := \iota \mid \top \mid A \times B \mid A \Rightarrow B \mid \Diamond A \qquad\qquad \textit{Ctx} \quad \Gamma, \Delta := \cdot \mid \Gamma, A
$$

The type ι denotes an uninterpeted base type (i.e. a ground type with no specific operations), $\top$ denotes the unit type, $A \times B$ denotes product types, and $A \Rightarrow B$ denotes function types. A type $\Diamond A$ denotes a computation that performs some side-effects to return a value of type A. A context Γ is a list of formulas $A_1, A_2, ..., A_n$ and $\cdot$ denotes the empty context.

The terms, typing rules and equational theory of λ_{LL} are defined in Figure 1. The judgements $\Gamma \vdash t : A$ define intrinsically well-typed terms of λ_{LL} and judgements $\Gamma \vdash t \sim t' : A$ define well-typed equations. We define well-typed (and scoped) variables using de Bruijn indices as judgments $\Gamma \vdash_{\text{VAR}} v : A$ with constructs zero and succ. For the sake of readability we will write terms using named variables as $\lambda\, x.\, \lambda\, y.\, x$ instead of $\lambda\, \lambda\, (\mathsf{var}\, (\mathsf{succ}\, \mathsf{zero}))$.

Admissibility. The notation $t[u]$ denotes the substitution of term u in t for the variable zero, and the notation $\mathit{wk}\, t$ denotes the weakening of a term $\Gamma \vdash t : A$ by embedding it into a larger context $\Gamma \leq \Gamma'$ as $\Gamma' \vdash \mathit{wk}\, t : A$. Both of the following rules are admissible in the calculus.

$$\textsc{Subst}\ \frac{\Gamma, A \vdash t : B \qquad \Gamma \vdash u : A}{\Gamma \vdash t[u] : B} \qquad\qquad \textsc{Wk}\ \frac{\Gamma \vdash t : A \qquad \Gamma \leq \Gamma'}{\Gamma' \vdash \mathit{wk}\, t : A}$$

Crucially, the modal axioms S, R and J are derivable in λ_{LL} as shown below. We write "$\mathsf{let}_{\text{LL}}\, x = t \text{ in } u$" with an explicit variable binding instead of "$\mathsf{let}_{\text{LL}}\, t\, u$" with de Bruijn indices.

- $\cdot \vdash \lambda\, x.\, \mathsf{let}_{\text{LL}}\, y = \mathsf{snd}\, x \text{ in } (\mathsf{return}_{\text{LL}}\, (\mathsf{pair}\, (\mathsf{fst}\, x)\, y)) : A \times \Diamond B \Rightarrow \Diamond(A \times B)$
- $\cdot \vdash \lambda\, x.\, \mathsf{return}_{\text{LL}}\, x : A \Rightarrow \Diamond A$
- $\cdot \vdash \lambda\, x.\, \mathsf{let}_{\text{LL}}\, y = x \text{ in } (\mathsf{let}_{\text{LL}}\, z = y \text{ in } \mathsf{return}_{\text{LL}}\, z) : \Diamond\Diamond A \Rightarrow \Diamond A$

Semantics. The semantics of LL is given using categories. A categorical model of λ_{LL} is a cartesian-closed category equipped with a strong monad $\Diamond$ (defined in Appendix A). Given a categorical model $\mathcal{C}$ of λ_{LL}, we interpret types and contexts in λ_{LL} as $\mathcal{C}$-objects and terms $\Gamma \vdash t : A$ as $\mathcal{C}$-morphisms $[\![t]\!] : [\![\Gamma]\!] \to [\![A]\!]$ by induction on types and terms respectively. The interpretation of the term constructs $\mathsf{return}_{\text{LL}}$ and let_{LL} (and in turn the modal axioms S, R and J) in a model of λ_{LL} is given by the structure of the strong monad $\Diamond$. We refer the reader to the accompanying Agda mechanization for further details.

▶ **Proposition 2** (Categorical semantics for λ_{LL})**.** *Given two terms t, u in λ_{LL}, $\Gamma \vdash t \sim u : A$ if and only if for all categorical models $\mathcal{C}$ of λ_{LL} $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \to [\![A]\!]$ in $\mathcal{C}$.*

Proof. Follows by induction on the judgment $\Gamma \vdash t \sim u : A$ in one direction, and by a term model construction (see for e.g., [13, Section 3.2]) in the converse. ◀

2.3 Sublogics SL, SRL and SJL and corresponding lambda calculi

We now define the three new sublogics of LL of interest in this paper, namely SL, SRL and SJL, by specifying their respective Hilbert-style axiomatisation and possible-world semantics. The logic SL can be axiomatised by extending the usual axioms and rules of IPL with the modal axiom S. Furthermore, we axiomatise:

- the logic SRL by extending SL with axiom R
- the logic SJL by extending SL with axiom J
- the logic LL by extending SL with axioms R and J (as defined previously)

The semantics for SL, SRL and SJL is given as before for LL by restricting the definitions of frames. An SL-frame $F = (W, R_i, R_m)$ is a triple that consists of a set W of worlds, a reflexive transitive relation R_i, and a relation R_m (that need not be reflexive or transitive), satisfying the forward confluence and inclusion conditions. Furthermore, an SL-frame is

- an SRL-frame when R_m is reflexive

$$\textsc{Var-Zero}\quad \Gamma, A \vdash_{\text{VAR}} \mathsf{zero} : A \qquad \textsc{Var-Succ}\quad \frac{\Gamma \vdash_{\text{VAR}} v : A}{\Gamma, B \vdash_{\text{VAR}} \mathsf{succ}\, v : A} \qquad \textsc{Var}\quad \frac{\Gamma \vdash_{\text{VAR}} v : A}{\Gamma \vdash \mathsf{var}\, v : A}$$

$$\top\text{-}\textsc{Intro}\quad \Gamma \vdash \mathsf{unit} : \top \qquad \times\text{-}\textsc{Intro}\quad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash \mathsf{pair}\, t\, u : A \times B} \qquad \times\text{-}\textsc{Elim-1}\quad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathsf{fst}\, t : A} \qquad \times\text{-}\textsc{Elim-2}\quad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathsf{snd}\, t : B}$$

$$\Rightarrow\text{-}\textsc{Intro}\quad \frac{\Gamma, A \vdash t : B}{\Gamma \vdash \lambda t : A \Rightarrow B} \qquad \Rightarrow\text{-}\textsc{Elim}\quad \frac{\Gamma \vdash t : A \Rightarrow B \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{app}\, t\, u : B} \qquad \text{LL}/\Diamond\text{-}\textsc{Return}\quad \frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{return}_{\text{LL}}\, t : \Diamond A}$$

$$\text{LL}/\Diamond\text{-}\textsc{Let}\quad \frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : \Diamond B}{\Gamma \vdash \mathsf{let}_{\text{LL}}\, t\, u : \Diamond B} \qquad \top\text{-}\eta\quad \frac{\Gamma \vdash t : \top}{\Gamma \vdash t \sim \mathsf{unit} : \top} \qquad \times\text{-}\eta\quad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash t \sim \mathsf{pair}\,(\mathsf{fst}\, t)\,(\mathsf{snd}\, t) : A \times B}$$

$$\times\text{-}\beta_1\quad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash \mathsf{fst}\,(\mathsf{pair}\, t\, u) \sim t : A} \qquad \times\text{-}\beta_2\quad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash \mathsf{snd}\,(\mathsf{pair}\, t\, u) \sim u : B}$$

$$\Rightarrow\text{-}\eta\quad \frac{\Gamma \vdash t : A \Rightarrow B}{\Gamma \vdash t \sim \lambda\,(\mathsf{app}\,(\mathit{wk}\, t)\,(\mathsf{var}\,\mathsf{zero})) : A \Rightarrow B} \qquad \Rightarrow\text{-}\beta\quad \frac{\Gamma, A \vdash t : B \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{app}\,(\lambda\, t)\, u \sim t[u] : B}$$

$$\text{LL}/\Diamond\text{-}\beta\quad \frac{\Gamma \vdash t : A \qquad \Gamma, A \vdash u : \Diamond B}{\Gamma \vdash \mathsf{let}_{\text{LL}}\,(\mathsf{return}_{\text{LL}}\, t)\, u \sim u[t] : \Diamond B} \qquad \text{LL}/\Diamond\text{-}\eta\quad \frac{\Gamma \vdash t : \Diamond A}{\Gamma \vdash t \sim \mathsf{let}_{\text{LL}}\, t\,(\mathsf{return}_{\text{LL}}\,(\mathsf{var}\,\mathsf{zero})) : \Diamond A}$$

$$\text{LL}/\Diamond\text{-}\textsc{ass}\quad \frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : \Diamond B \qquad \Gamma, B \vdash u' : \Diamond C}{\Gamma \vdash \mathsf{let}_{\text{LL}}\,(\mathsf{let}_{\text{LL}}\, t\, u)\, u' \sim \mathsf{let}_{\text{LL}}\, t\,(\mathsf{let}_{\text{LL}}\, u\,(\mathit{wk}\, u')) : \Diamond C}$$

Figure 1 Well-typed terms and equational theory for λ_{LL}

- an SJL-frame when R_m is transitive
- an LL-frame when R_m is reflexive and transitive (as defined previously)

In the upcoming section, we will define corresponding lax modal lambda calculi for each of these sublogics (Section 3). We develop proof-relevant possible-world semantics for lax modal calculi and show the connection to categorical semantics by studying the properties of presheaf categories determined by proof-relevant frames (Section 4). We leverage this connection to then construct Normalization by Evaluation models for the calculi, and show as corollaries completeness and inadmissibility theorems (Section 5).

3 The Calculi λ_{SL}, λ_{SRL} and λ_{SJL}

The calculi λ_{SL}, λ_{SRL} and λ_{SJL} are defined as before with λ_{LL} as extensions of STLC with a unary type constructor $\Diamond$. The types and contexts of all four calculi are defined alike.

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : B}{\Gamma \vdash \mathsf{letmap}_{\mathrm{SL}}\, t\, u : \Diamond B}\ \text{SL}/\Diamond\text{-Letmap} \qquad \frac{\Gamma \vdash t : \Diamond A}{\Gamma \vdash t \sim \mathsf{letmap}_{\mathrm{SL}}\, t\, (\mathsf{var\,zero}) : \Diamond A}\ \text{SL}/\Diamond\text{-}\eta$$

$$\frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : B \qquad \Gamma, B \vdash u' : C}{\Gamma \vdash \mathsf{letmap}_{\mathrm{SL}}\, (\mathsf{letmap}_{\mathrm{SL}}\, t\, u)\, u' \sim \mathsf{letmap}_{\mathrm{SL}}\, t\, (u'[u]) : \Diamond C}\ \text{SL}/\Diamond\text{-}\beta$$

Figure 2 Well-typed terms and equational theory for λ_{SL} (omitting those of STLC)

The calculus λ_{SL}. The terms, typing rules and equational theory of the modal fragment of λ_{SL} is defined in Figure 2. λ_{SL} extends STLC with a construct $\mathsf{letmap}_{\mathrm{SL}}$ and two new equations SL/$\Diamond$-η and SL/$\Diamond$-β. Observe that the typing rule for $\mathsf{letmap}_{\mathrm{SL}}$ in λ_{SL} differs from $\mathsf{let}_{\mathrm{LL}}$ in λ_{LL}: a term $\mathsf{letmap}_{\mathrm{SL}}\, t\, u$ "maps" a term $\Gamma, A \vdash u : B$ over a term $\Gamma \vdash t : \Diamond A$ to yield a term of type $\Diamond B$ in context Γ. This difference disallows a derivation of axiom J, while allowing a derivation of axioms S as shown below:

$$\cdot \vdash \lambda\, x.\, \mathsf{letmap}_{\mathrm{SL}}\, y = \mathsf{snd}\, x \ \mathsf{in}\, (\mathsf{pair}\, (\mathsf{fst}\, x)\, y) : A \times \Diamond B \Rightarrow \Diamond(A \times B)$$

Since there is no counterpart to $\mathsf{return}_{\mathrm{LL}}$ in λ_{SL}, axiom R cannot be derived in λ_{SL}.

A categorical model of λ_{SL} is a cartesian-closed category equipped with a strong functor $\Diamond$ (that need not be a monad). Given a categorical model $\mathcal{C}$ of λ_{SL}, we interpret types and contexts in λ_{SL} as $\mathcal{C}$-objects and terms $\Gamma \vdash t : A$ as $\mathcal{C}$-morphisms $[\![t]\!] : [\![\Gamma]\!] \to [\![A]\!]$ as before with λ_{LL} by induction on types and terms respectively. The interpretation of the term construct $\mathsf{letmap}_{\mathrm{SL}}$ (and in turn the modal axiom S) is given by the tensorial strength of functor $\Diamond$, which gives us a morphism $X \times \Diamond Y \to \Diamond(X \times Y)$ for all objects X, Y in $\mathcal{C}$.

▶ **Proposition 3** (Categorical semantics for λ_{SL})**.** *Given two terms t, u in λ_{LL}, $\Gamma \vdash t \sim u : A$ if and only if for all categorical models $\mathcal{C}$ of λ_{SL} $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \to [\![A]\!]$ in $\mathcal{C}$.*

The calculus λ_{SRL}. The terms, typing rules and equational theory for the modal fragment of λ_{SRL} are defined in Figure 3. λ_{SRL} extends STLC with two constructs $\mathsf{return}_{\mathrm{SRL}}$ and $\mathsf{letmap}_{\mathrm{SRL}}$, and three new equations SRL/$\Diamond$-η, SRL/$\Diamond$-β_1 and SRL/$\Diamond$-β_2. Observe that the typing rule for $\mathsf{letmap}_{\mathrm{SRL}}$ is identical to $\mathsf{letmap}_{\mathrm{SL}}$ and axiom S can be derived in λ_{SRL} exactly as above in λ_{SL}:

$$\cdot \vdash \lambda\, x.\, \mathsf{letmap}_{\mathrm{SRL}}\, y = \mathsf{snd}\, x \ \mathsf{in}\, (\mathsf{pair}\, (\mathsf{fst}\, x)\, y) : A \times \Diamond B \Rightarrow \Diamond(A \times B)$$

Axiom R can as well be derived since the typing rule of $\mathsf{return}_{\mathrm{SRL}}$ is identical to $\mathsf{return}_{\mathrm{LL}}$.

$$\cdot \vdash \lambda\, x.\, \mathsf{return}_{\mathrm{SRL}}\, x : A \Rightarrow \Diamond A$$

Axiom J, on the other hand, cannot be derived in λ_{SRL}.

A categorical model of λ_{SRL} is a cartesian-closed category equipped with a strong pointed functor $\Diamond$. The term construct $\mathsf{letmap}_{\mathrm{SRL}}$ (and in turn axiom S) is interpreted in a model $\mathcal{C}$ of λ_{SRL} using the tensorial strength of functor $\Diamond$, as before with λ_{SL}. The interpretation of the term construct $\mathsf{return}_{\mathrm{SRL}}$ (and in turn axiom R) is given by the pointed structure of the functor $\Diamond$, which gives us a morphism $X \to \Diamond X$ for all objects X in $\mathcal{C}$.

▶ **Proposition 4** (Categorical semantics for λ_{SRL})**.** *Given two terms t, u in λ_{SRL}, $\Gamma \vdash t \sim u : A$ if and only if for all categorical models $\mathcal{C}$ of λ_{SRL} $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \to [\![A]\!]$ in $\mathcal{C}$.*

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{return}_{\mathrm{SRL}}\, t : \lozenge A}\ \text{SRL}/\lozenge\text{-Return} \qquad \frac{\Gamma \vdash t : \lozenge A \qquad \Gamma, A \vdash u : B}{\Gamma \vdash \mathsf{letmap}_{\mathrm{SRL}}\, t\, u : \lozenge B}\ \text{SRL}/\lozenge\text{-Letmap}$$

$$\frac{\Gamma \vdash t : \lozenge A}{\Gamma \vdash t \sim \mathsf{letmap}_{\mathrm{SRL}}\, t\, (\mathsf{var\ zero}) : \lozenge A}\ \text{SRL}/\lozenge\text{-}\eta$$

$$\frac{\Gamma \vdash t : \lozenge A \qquad \Gamma, A \vdash u : B \qquad \Gamma, B \vdash u' : C}{\Gamma \vdash \mathsf{letmap}_{\mathrm{SRL}}\, (\mathsf{letmap}_{\mathrm{SRL}}\, t\, u)\, u' \sim \mathsf{letmap}_{\mathrm{SRL}}\, t\, (u'[u]) : \lozenge C}\ \text{SRL}/\lozenge\text{-}\beta_1$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma, A \vdash u : B}{\Gamma \vdash \mathsf{letmap}_{\mathrm{SRL}}\, (\mathsf{return}_{\mathrm{SRL}}\, t)\, u \sim \mathsf{return}_{\mathrm{SRL}}\, (u[t]) : \lozenge B}\ \text{SRL}/\lozenge\text{-}\beta_2$$

Figure 3 Well-typed terms and equational theory for λ_{SRL} (omitting those of STLC)

The calculus λ_{SJL}. The terms, typing rules and equational theory are defined for the modal fragment of λ_{SJL} in Figure 4. λ_{SJL} extends STLC with two constructs $\mathsf{letmap}_{\mathrm{SJL}}$ and $\mathsf{let}_{\mathrm{SJL}}$, and five equations SJL/$\lozenge$-η, SJL/$\lozenge$-β_1, SJL/$\lozenge$-η_2, SJL/$\lozenge$-com and SJL/$\lozenge$-ass.

Observe that the typing rule for $\mathsf{letmap}_{\mathrm{SJL}}$ is identical to $\mathsf{letmap}_{\mathrm{SL}}$ and axiom S can be derived in λ_{SJL} exactly as before in λ_{SRL} and λ_{SL}:

$$\cdot \vdash \lambda\, x.\, \mathsf{letmap}_{\mathrm{SJL}}\, y = \mathsf{snd}\, x \ \mathsf{in}\, (\mathsf{pair}\, (\mathsf{fst}\, x)\, y) : A \times \lozenge B \Rightarrow \lozenge(A \times B)$$

Axiom J can be derived in λ_{SJL} using a combination of $\mathsf{letmap}_{\mathrm{SJL}}$ and $\mathsf{let}_{\mathrm{SJL}}$ as:

$$\cdot \vdash \lambda\, x.\, \mathsf{let}_{\mathrm{SJL}}\, y = x \ \mathsf{in}\, (\mathsf{letmap}_{\mathrm{SJL}}\, z = y \ \mathsf{in}\, z) : \lozenge\lozenge A \Rightarrow \lozenge A$$

Axiom R, however, cannot be derived in λ_{SJL}.

A categorical model of λ_{SJL} is a cartesian-closed category equipped with a strong semi-monad $\lozenge$. We interpret the term construct $\mathsf{letmap}_{\mathrm{SJL}}$ (and in turn axiom S) in a categorical model $\mathcal{C}$ of λ_{SJL}, using the tensorial strength of functor $\lozenge$ as before with λ_{SL} and λ_{SRL}. The interpretation of the term construct $\mathsf{let}_{\mathrm{SJL}}$ (and in turn axiom J) is given by the semimonad structure of functor $\lozenge$, which gives us a morphism $\lozenge\lozenge X \to \lozenge X$ for all objects X in $\mathcal{C}$.

▶ **Proposition 5** (Categorical semantics for λ_{SJL}). *Given two terms t, u in λ_{SJL}, $\Gamma \vdash t \sim u : A$ if and only if for all categorical models $\mathcal{C}$ of λ_{SJL} $[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \to [\![A]\!]$ in $\mathcal{C}$.*

4 Proof-relevant possible-world semantics

Possible-world semantics is typically given for a logic in a classical meta-language using sets and relations, as in Section 2. In this section, we are concerned with possible-world semantics for lambda calculi, for which we will instead work in a constructive dependent type-theory based on Agda. We will use a type X : *Type* in place of a set X and values $x : X$ in place of elements $x \in X$. The arrow $\to$ denotes functions, and quantifications $\forall x$ and Σ_x denote universal and existential quantification respectively, where $x : X$ is an element of some type X : *Type* that is left implicit. A value of type $\forall x.\, P(x)$ for some predicate $P : X \to$ *Type* is a function $\lambda x.\, p$ with $p : P(x)$. A value of type $\Sigma_x.\, P(x)$ is a tuple (x, p), but we will leave the witness x implicit at times and write $(\_, p)$ or simply p for brevity.

$$\textsc{SJL}/\Diamond\text{-Letmap}\quad \frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : B}{\Gamma \vdash \mathsf{letmap}_{\mathrm{SJL}}\, t\, u : \Diamond B}$$

$$\textsc{SJL}/\Diamond\text{-Let}\quad \frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : \Diamond B}{\Gamma \vdash \mathsf{let}_{\mathrm{SJL}}\, t\, u : \Diamond B}$$

$$\textsc{SJL}/\Diamond\text{-}\eta\quad \frac{\Gamma \vdash t : \Diamond A}{\Gamma \vdash t \sim \mathsf{letmap}_{\mathrm{SJL}}\, t\, (\mathsf{var}\ \mathsf{zero}) : \Diamond A}$$

$$\textsc{SJL}/\Diamond\text{-}\beta_1\quad \frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : B \qquad \Gamma, B \vdash u' : C}{\Gamma \vdash \mathsf{letmap}_{\mathrm{SJL}}\, (\mathsf{letmap}_{\mathrm{SJL}}\, t\, u)\, u' \sim \mathsf{letmap}_{\mathrm{SJL}}\, t\, (u'[u]) : \Diamond C}$$

$$\textsc{SJL}/\Diamond\text{-}\beta_2\quad \frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : B \qquad \Gamma, B \vdash u' : \Diamond C}{\Gamma \vdash \mathsf{let}_{\mathrm{SJL}}\, (\mathsf{letmap}_{\mathrm{SJL}}\, t\, u)\, u' \sim \mathsf{let}_{\mathrm{SJL}}\, t\, (u'[u]) : \Diamond C}$$

$$\textsc{SJL}/\Diamond\text{-com}\quad \frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : \Diamond B \qquad \Gamma, B \vdash u' : C}{\Gamma \vdash \mathsf{letmap}_{\mathrm{SJL}}\, (\mathsf{let}_{\mathrm{SJL}}\, t\, u)\, u' \sim \mathsf{let}_{\mathrm{SJL}}\, t\, (\mathsf{letmap}_{\mathrm{SJL}}\, u\, (\mathit{wk}\, u')) : \Diamond C}$$

$$\textsc{SJL}/\Diamond\text{-ass}\quad \frac{\Gamma \vdash t : \Diamond A \qquad \Gamma, A \vdash u : \Diamond B \qquad \Gamma, B \vdash u' : \Diamond C}{\Gamma \vdash \mathsf{let}_{\mathrm{SJL}}\, (\mathsf{let}_{\mathrm{SJL}}\, t\, u)\, u' \sim \mathsf{let}_{\mathrm{SJL}}\, t\, (\mathsf{let}_{\mathrm{SJL}}\, u\, (\mathit{wk}\, u')) : \Diamond C}$$

Figure 4 Well-typed terms and equational theory for λ_{SJL} (omitting those of STLC)

Semantics for λ_{SL}. A proof-relevant λ_{SL}-frame $F = (W, R_i, R_m)$ is a triple that consists of a type W : *Type* of worlds and two proof-relevant relations $R_i, R_m \colon W \to W \to \mathit{Type}$ with

- functions $\mathit{refl}_i : \forall w.\, w\ R_i\ w$ and $\mathit{trans}_i : \forall w, w', w''.\, w\ R_i\ w' \to w'\ R_i\ w'' \to w\ R_i\ w''$ respectively proving the reflexivity and transitivity of R_i such that
  - $\mathit{trans}_i\, \mathit{refl}_i\, i = i$ and $\mathit{trans}_i\, i\, \mathit{refl}_i = i$
  - $\mathit{trans}_i\, (\mathit{trans}_i\, i\, i')\, i'' = \mathit{trans}_i\, i\, (\mathit{trans}_i\, i'\, i'')$
- function $\mathit{factor} : \forall w, w', v.\, w\ R_i\ w' \to w\ R_m\ v \to \Sigma_{v'}.\, (w'\ R_m\ v' \times v\ R_i\ v')$ such that
  - $\mathit{factor}\, \mathit{refl}_i\, m = (m, \mathit{refl}_i)$
  - $\mathit{factor}\, (\mathit{trans}_i\, i_1\, i_2)\, m = (m'_2, (\mathit{trans}_i\, i'_1\, i'_2))$
    where $(i'_1, m'_1) = \mathit{factor}\, i_1\, m$ and $(i'_2, m'_2) = \mathit{factor}\, i_2\, m'_1$.
- function $\mathit{incl} : \forall w, v.\, w\ R_m\ v \to w\ R_i\ v$, such that
  - $\mathit{trans}_i\, i\, (\mathit{incl}\, m') = \mathit{trans}_i\, (\mathit{incl}\, m)\, i'$, where $(i', m') = \mathit{factor}\, i\, m$

The function refl_i and trans_i are the proof-relevant encoding of reflexivity and transitivity of R_i respectively. These functions are subject to the accompanying coherence laws, which state that the proof computed by refl_i must be the unit of trans_i, i.e. R_i must form a category $\mathcal{W}_i$. The coherence laws facilitate a sound interpretation of λ_{SL}'s equational theory.

The functions *factor* and *incl* are proof-relevant encodings of the forward confluence (${R_i}^{-1}; R_m \subseteq R_m; {R_i}^{-1}$) and inclusion ($R_m \subseteq R_i$) conditions respectively. Given a proof of $w\ R_i\ w'$ (i.e. $w'\ {R_i}^{-1}\ w$) and $w\ R_m\ v$, *factor* returns a pair of proofs for some world v':

$w' \; R_m \; v'$ and $v \; R_i \; v'$ (i.e. $v' \; R_i^{-1} \; v$). Similarly, given a proof of $w \; R_m \; v$, *incl* returns a proof of $w \; R_i \; v$. These functions are also accompanied by the stated coherence laws.

The proof-relevant relation R_i in a λ_{SL}-frame determines a category $\mathcal{W}_i$ whose objects are given by worlds and morphisms by proofs of R_i, with refl_i witnessing the identity morphisms and trans_i witnessing the composition of morphisms. This determines a category $\widehat{\mathcal{W}_i}$ of covariant presheaves indexed by $\mathcal{W}_i$. The objects in the category $\widehat{\mathcal{W}_i}$ are presheaves and the morphisms are natural transformations. A presheaf P is given by a family of meta-language types $P_w : \mathit{Type}$ indexed by worlds $w : W$, accompanied by "weakening" functions $\mathit{wk} : \forall w, w'.\, w \; R_i \; w' \to P_w \to P_{w'}$ subject to certain conditions. A natural transformation $f : P \dot{\to} Q$ is a family of functions $\forall w.\, P_w \to Q_w$ subject to a naturality condition.

▶ **Proposition 6** (◊ Strong Functor). *The presheaf category $\widehat{\mathcal{W}_i}$ determined by a λ_{SL}-frame exhibits a strong endofunctor $(\Diamond P)_w = \Sigma_v.\, w \; R_m \; v \times P_v$ for some world w and presheaf P.*

Proof. The function *factor* defines the presheaf action of $\Diamond P$ and the coherence conditions on *factor* (e.g., $\mathit{factor}\; \mathit{refl}_i\; m = (m, \mathit{refl}_i)$) prove the presheaf conditions. The functorial action of $\Diamond$ on a natural transformation $f : \forall w.\, P_w \to Q_w$ is defined by applying f at the world v witnessing the Σ quantification. The functorial laws of $\Diamond$ follow immediately and the strength of $\Diamond$ is given by the function *incl* and the coherence laws that accompany it. ◀

Propositions 3 and 6 give us that $\widehat{\mathcal{W}_i}$ is a categorical model of λ_{SL}. For clarity, we elaborate on this consequence by giving a direct interpretation of λ_{SL} in $\widehat{\mathcal{W}_i}$.

A proof-relevant possible-world *model* $M = (F, V)$ couples a proof-relevant frame F with a valuation function V that assigns to a base type ι a presheaf $V_\iota : W \to \mathit{Type}$. Given such a model, the types in λ_{SL} are interpreted as presheaves, i.e. we interpret a type A as a family $[\![A]\!]_w : \mathit{Type}$ indexed by an arbitrary world $w : W$—as shown on the left below.

$$
\begin{aligned}
[\![\iota]\!]_w &= V_{\iota,w} \\
[\![A \times B]\!]_w &= [\![A]\!]_w \times [\![B]\!]_w \\
[\![A \Rightarrow B]\!]_w &= \forall w'.\, w \; R_i \; w' \to [\![A]\!]_{w'} \to [\![B]\!]_{w'} \\
[\![\Diamond A]\!]_w &= \textstyle\sum_v .\, w \; R_m \; v \times [\![A]\!]_v
\end{aligned}
$$

$$
\begin{aligned}
&[\![-]\!] : \Gamma \vdash A \to (\forall w.\, [\![\Gamma]\!]_w \to [\![A]\!]_w) \\
&[\![\mathsf{var}\, v]\!]\, \gamma = \mathit{lookup}\, v\, \gamma \\
&[\![\mathsf{unit}]\!]\, \gamma = () \\
&[\![\mathsf{pair}\, t\, u]\!]\, \gamma = ([\![t]\!]\, \gamma, [\![u]\!]\, \gamma) \\
&[\![\mathsf{fst}\, t]\!]\, \gamma = \pi_1([\![t]\!]\, \gamma) \\
&[\![\mathsf{snd}\, t]\!]\, \gamma = \pi_2([\![t]\!]\, \gamma) \\
&[\![\lambda\, t]\!]\, \gamma = \lambda i.\, \lambda a.\, [\![t]\!]\, (\mathit{wk}\, i\, \gamma, a) \\
&[\![\mathsf{app}\, t\, u]\!]\, \gamma = ([\![t]\!]\, \gamma)\; \mathit{refl}_i\; ([\![u]\!]\, \gamma) \\
&[\![\mathsf{letmap}_{\mathrm{SL}}\, t\, u]\!]\, \gamma = (m, [\![u]\!]\, (\mathit{wk}\, (\mathit{incl}\, m)\, \gamma, x)) \\
&\qquad \text{where } (m : w \; R_m \; v, x : [\![A]\!]_v) = [\![t]\!]\, \gamma
\end{aligned}
$$

The interpretation of the base type ι is given by the valuation function V, and the unit, product and function types are interpreted as usual using their semantic counterparts. We interpret the $\Diamond$ modality using the proof-relevant quantifier $\sum$: the interpretation of a type $\Diamond A$ at a world w is given by the interpretation of A at some modal future world v along with a proof of $w \; R_m \; v$ witnessing the connection from w to v via R_m. The typing contexts are interpreted as usual by taking the cartesian product of presheaves.

The terms in λ_{SL} are interpreted as natural transformations by induction on the typing judgment. Interpretation of STLC terms follows the usual routine: we interpret variables by projecting the environment $\gamma : [\![\Gamma]\!]_w$ using a function *lookup*, the unit and pair constructs (unit, pair, fst, snd) with their semantic counterparts ($()$, $(-,-)$, π_1, π_2), and the function constructs (λ,app) with semantic function abstraction and application. The interesting case is that of $\mathsf{letmap}_{\mathrm{SL}}$: given terms $\Gamma \vdash t : \Diamond A$ and $\Gamma, A \vdash u : B$, and an environment $\gamma : [\![\Gamma]\!]_w$, we must produce an element of type $[\![\Diamond B]\!]_w = \sum_v .\, w \; R_m \; v \times [\![B]\!]_v$. Recursively interpreting t

gives us a pair $(m : w\ R_m\ v, x : [\![A]\!]_v)$, using the former of which we transport γ along R_m to the world v, as $\mathit{wk}_\Gamma\,(\mathit{incl}\,m)\,\gamma : [\![\Gamma]\!]_v$, which is in turn used to recursively interpret u, thus obtaining the desired element of type $[\![B]\!]_v$.

Semantics for λ_{SRL}. A proof-relevant λ_{SRL}-frame (W, R_i, R_m) is a λ_{SL}-frame that exhibits:

- function $\mathit{refl}_m : \forall w.\, w\ R_m\ w$, such that
  - $\mathit{factor}\ i\ \mathit{refl}_m = (\mathit{refl}_m, i)$
  - $\mathit{incl}\ \mathit{refl}_m = \mathit{refl}_i$

▶ **Proposition 7** (◊ Strong Pointed). *The strong functor ◊ on the category of presheaves $\widehat{\mathcal{W}_i}$ determined by a λ_{SRL}-frame is strong pointed.*

Proof. To show that ◊ is pointed, we define $\mathit{point} : P \dot{\to} \Diamond P$ using function refl_m, and then use the coherence law $\mathit{incl}\ \mathit{refl}_m = \mathit{refl}_i$ to show that point is a strong natural transformation. ◀

The interpretation of the modal fragment of λ_{SRL} can be explicitly given as:

$$\begin{aligned}
[\![\mathsf{return}_{\mathrm{SRL}}\ t\ ]\!]\,\gamma &= (\mathit{refl}_m, [\![t]\!]\,\gamma) \\
[\![\mathsf{letmap}_{\mathrm{SRL}}\ t\ u]\!]\,\gamma &= (m, [\![u]\!]\,(\mathit{wk}\,(\mathit{incl}\,m)\,\gamma, x)) \\
&\quad \text{where } (m : w\ R_m\ v, x : [\![A]\!]_v) = [\![t]\!]\,\gamma
\end{aligned}$$

Semantics for λ_{SJL}. A proof-relevant λ_{SJL}-frame (W, R_i, R_m) is a λ_{SL}-frame that exhibits:

- function $\mathit{trans}_m : \forall u, v, w.\ u\ R_m\ v \to v\ R_m\ w \to u\ R_m\ w$, such that
  - $\mathit{factor}\ i\,(\mathit{trans}_m\, m_1\, m_2) = (\mathit{trans}_m\, m_1'\, m_2', i_2')$
    where $(i_1', m_1') = \mathit{factor}\ i\ m_1$ and $(i_2', m_2') = \mathit{factor}\ i_1'\ m_1'$.
  - $\mathit{trans}_m\,(\mathit{trans}_m\, m_1\, m_2)\, m_3 = \mathit{trans}_m\, m_1\,(\mathit{trans}_m\, m_2\, m_3)$
  - $\mathit{incl}\,(\mathit{trans}_m\, m_1\, m_2) = \mathit{trans}_i\,(\mathit{incl}\, m_1)\,(\mathit{incl}\, m_2)$

▶ **Proposition 8** (◊ Strong Semimonad). *The strong functor ◊ on the category of presheaves $\widehat{\mathcal{W}_i}$ determined by a λ_{SJL}-frame is a strong semimonad.*

Proof. We define $\mu : \Diamond\Diamond P \dot{\to} \Diamond P$ using the function trans_m to show ◊ is a semimonad, and then use the coherence law $\mathit{incl}\,(\mathit{trans}_m\, m_1\, m_2) = \mathit{trans}_i\,(\mathit{incl}\, m_1)\,(\mathit{incl}\, m_2)$ to show that μ is a strong natural transformation—giving us that μ is a strong semimonad. ◀

The interpretation of the modal fragment of λ_{SJL} can be explicitly given as:

$$\begin{aligned}
[\![\mathsf{letmap}_{\mathrm{SJL}}\ t\ u]\!]\,\gamma &= (m, [\![u]\!]\,(\mathit{wk}\,(\mathit{incl}\,m)\,\gamma, x)) \\
&\quad \text{where } (m : w\ R_m\ v, x : [\![A]\!]_v) = [\![t]\!]\,\gamma \\
[\![\mathsf{let}_{\mathrm{SJL}}\ t\ u\ ]\!]\,\gamma &= (\mathit{trans}_m\, m\, m', y) \\
&\quad \text{where } (m : w\ R_m\ v, x : [\![A]\!]_v) = [\![t]\!]\,\gamma \\
&\qquad\quad (m' : v\ R_m\ v', y : [\![B]\!]_{v'}) = [\![u]\!]\,(\mathit{wk}\,(\mathit{incl}\,m)\,\gamma, x)
\end{aligned}$$

Semantics for λ_{LL}. A proof-relevant λ_{LL}-frame $F = (W, R_i, R_m)$ is both an λ_{SRL}-frame and λ_{SJL}-frame that further exhibits the unit laws $\mathit{trans}_m\ \mathit{refl}_m\ m = m$ and $\mathit{trans}_m\ m\ \mathit{refl}_m = m$. That is, proofs of R_m now form a category $\mathcal{W}_m$ with a functor $\mathcal{W}_m \to \mathcal{W}_i$ given by function incl.

▶ **Proposition 9** (◊ Strong Monad). *The strong functor ◊ on the category of presheaves $\widehat{\mathcal{W}_i}$ determined by a λ_{LL}-frame is a strong monad.*

Proof. We apply Propositions 6–8 to show that the functor ◊ is a strong pointed semimonad. We then use the unit laws of the category $\mathcal{W}_m$ to prove the unit laws of the monad ◊. ◀

$$\frac{\Gamma \vdash_{\mathrm{VAR}} v : A}{\Gamma \vdash_{\mathrm{NE}} \mathsf{var}\, v : A}\ \textsc{Ne/Var} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \iota}{\Gamma \vdash_{\mathrm{NF}} \mathsf{up}\, n : \iota}\ \textsc{Nf/Up} \qquad \frac{}{\Gamma \vdash_{\mathrm{NF}} \mathsf{unit} : \top}\ \textsc{Nf/Unit} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : A \times B}{\Gamma \vdash_{\mathrm{NE}} \mathsf{fst}\, n : A}\ \textsc{Ne/}{\times}\textsc{-Elim-1}$$

$$\frac{\Gamma \vdash_{\mathrm{NE}} n : A \times B}{\Gamma \vdash_{\mathrm{NE}} \mathsf{snd}\, n : B}\ \textsc{Ne/}{\times}\textsc{-Elim-2} \qquad \frac{\Gamma \vdash_{\mathrm{NF}} n : A \qquad \Gamma \vdash_{\mathrm{NF}} m : B}{\Gamma \vdash_{\mathrm{NF}} \mathsf{pair}\, n\, m : A \times B}\ \textsc{Nf/}{\times}\textsc{-Intro} \qquad \frac{\Gamma, A \vdash_{\mathrm{NF}} n : B}{\Gamma \vdash_{\mathrm{NF}} \lambda n : A \Rightarrow B}\ \textsc{Nf/}{\Rightarrow}\textsc{-Intro}$$

$$\frac{\Gamma \vdash_{\mathrm{NE}} n : A \Rightarrow B \qquad \Gamma \vdash_{\mathrm{NF}} m : A}{\Gamma \vdash_{\mathrm{NE}} \mathsf{app}\, n\, m : B}\ \textsc{Ne/}{\Rightarrow}\textsc{-Elim} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \qquad \Gamma, A \vdash_{\mathrm{NF}} m : B}{\Gamma \vdash_{\mathrm{NF}} \mathsf{letmap}_{\mathrm{SL}}\, n\, m : \Diamond B}\ \textsc{NF/}\Diamond\textsc{-Letmap/SL}$$

Figure 5 Neutral terms and Normal forms for λ_{SL}

The interpretation of the modal fragment of λ_{LL} can be explicitly given as:

$$\begin{aligned}
&[\![\mathsf{return}_{\mathrm{LL}}\, t]\!]\,\gamma = (\mathit{refl}_m, [\![t]\!]\,\gamma)\\
&[\![\mathsf{let}_{\mathrm{LL}}\, t\, u\ ]\!]\,\gamma = (\mathit{trans}_m\, m\, m', y)\\
&\qquad \text{where } (m : w\ R_m\ v, x : [\![A]\!]_v) = [\![t]\!]\,\gamma\\
&\qquad\qquad (m' : v\ R_m\ v', y : [\![B]\!]_{v'}) = [\![u]\!]\,(\mathit{wk}\,(\mathit{incl}\, m)\,\gamma, x)
\end{aligned}$$

▶ **Theorem 10** (Soundness of proof-relevant possible-world semantics)**.** *For any two terms $\Gamma \vdash t, u : A$ in $\lambda_{SL}/\lambda_{SRL}/\lambda_{SJL}/\lambda_{LL}$, if $\Gamma \vdash t \sim u : A$ then $[\![t]\!] = [\![u]\!]$ for an arbitrary proof-relevant possible-world model determined by the respective $\lambda_{SL}/\lambda_{SRL}/\lambda_{SJL}/\lambda_{LL}$-frames.*

Proof. Applying Propositions 6–9 respectively to Propositions 2–5 gives us that the category $\widehat{\mathcal{W}_i}$ determined by a $\lambda_{\mathrm{SL}}/\lambda_{\mathrm{SRL}}/\lambda_{\mathrm{SJL}}/\lambda_{\mathrm{LL}}$-frame is a categorical model of the respective calculus. As a result, we get the soundness of the equational theory for possible-world models via soundness of the equational theory for categorical models. ◀

5 Normalization, Completeness and Inadmissibility results

Catarina Coquand [14, 15] proved normalization for STLC in the proof assistant Alf [28] by constructing an instance of Mitchell and Moggi's proof-relevant possible-world semantics. This model-based approach to normalization, known as Normalization by Evaluation (NbE) [9, 8], dispenses with tedious syntactic reasoning that typically complicate normalization proofs. In this section, we extend Coquand's result to lax modal lambda calculi and observe corollaries including completeness and inadmissibility of irrelevant modal axioms.

The objective of NbE is to define a function $\mathit{norm} : \Gamma \vdash A \to \Gamma \vdash_{\mathrm{NF}} A$, assigning a normal form to every term in the calculus. We write $\Gamma \vdash A$ to denote all terms $\Gamma \vdash t : A$ and $\Gamma \vdash_{\mathrm{NF}} A$ to denote all normal forms $\Gamma \vdash_{\mathrm{NF}} n : A$. Normal forms are defined as usual alongside judgements $\Gamma \vdash_{\mathrm{NE}} n : A$ denoting *neutral* terms, which can be understood as "straight-forward" inferences that do not involve introduction rules.

To define *norm* for λ_{SL}, we construct a possible-world model (N, V), known as the NbE model, with a λ_{SL}-frame $N = (\mathit{Ctx}, \leq, \lhd_{\mathrm{SL}})$ consisting of contexts for worlds, the weakening relation $\leq$ for R_i and the accessibility relation $\lhd_{\mathrm{SL}}$ for R_m. The valuation is given by neutral terms as $V_{\iota,\Gamma} = \Gamma \vdash_{\mathrm{NE}} \iota$. The relations $\leq$ and $\lhd_{\mathrm{SL}}$ on contexts are defined inductively as follows. The latter definition states that $\Gamma \lhd_{\mathrm{SL}} \Delta$ if and only if $\Delta \equiv \Gamma, A$ for some type A such that there exists a neutral term $\Gamma \vdash n : \Diamond A$.

$$\frac{}{\mathit{base} : \cdot \leq \cdot} \quad \frac{i : \Gamma \leq \Gamma'}{\mathit{drop}_A\, i : \Gamma \leq \Gamma', A} \quad \frac{i : \Gamma \leq \Gamma'}{\mathit{keep}_A\, i : \Gamma, A \leq \Gamma', A} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A}{\mathsf{single}\, n : \Gamma \lhd_{\mathrm{SL}} \Gamma, A}$$

The proof-relevant relation $\lhd_{\mathrm{SL}}$ is neither reflexive nor transitive, but is included in the $\leq$ relation since we can define a function *incl* : $\forall \Gamma, \Delta.\, \Gamma \lhd_{\mathrm{SL}} \Delta \to \Gamma \leq \Delta$. We can also show that the λ_{SL}-frame N satisfies the forward confluence condition by defining a function *factor* : $\forall \Gamma, \Gamma', \Delta.\, \Gamma \leq \Gamma' \to \Gamma \lhd_{\mathrm{SL}} \Delta \to \exists \Delta'.\, (\Gamma' \lhd_{\mathrm{SL}} \Delta' \times \Delta \leq \Delta')$.

By construction, we obtain an interpretation of terms $[\![-]\!] : \Gamma \vdash A \to (\forall \Delta.\, [\![\Gamma]\!]_\Delta \to [\![A]\!]_\Delta)$ in the NbE model as an instance of the generic interpreter for an arbitrary possible-world model (Section 4). This model exhibits two type-indexed functions characteristic of NbE models known as *reify* and *reflect*, which are defined for the modal fragment as follows:

$$\begin{aligned}
&\mathit{reify}_A : \forall \Gamma.\, [\![A]\!]_\Gamma \to \Gamma \vdash_{\mathrm{NF}} A \\
&\ldots \\
&\mathit{reify}_{\Diamond A;\Gamma}\, ((\mathsf{single}\, n : \Gamma \lhd_{\mathrm{SL}} \Gamma, B), x : [\![A]\!]_{\Gamma,B}) = \mathsf{letmap}_{\mathrm{SL}}\, n\, (\mathit{reify}_{A;(\Gamma,B)}\, x)
\end{aligned}$$

$$\begin{aligned}
&\mathit{reflect}_A : \forall \Gamma.\, \Gamma \vdash_{\mathrm{NE}} A \to [\![A]\!]_\Gamma \\
&\ldots \\
&\mathit{reflect}_{\Diamond A;\Gamma}\, n = (\mathsf{single}\, n, \mathit{reflect}_{A;(\Gamma,A)}\, \mathsf{var}\, \mathsf{zero})
\end{aligned}$$

The function *reify* is a type-indexed natural transformation, which for the case of type $\Diamond A$ in some context Γ, is given as argument an element of type $[\![\Diamond A]\!]_\Gamma$, which is $\Sigma_\Delta.\Gamma \lhd_{\mathrm{SL}} \Delta \times [\![A]\!]_\Delta$. The first component gives us a neutral $\Gamma \vdash_{\mathrm{NE}} n : B$, and recursively reifying the second component gives us a normal form of $\Gamma, B \vdash_{\mathrm{NF}} \mathit{reify}_{A;(\Gamma,B)}\, x : A$. We use these to construct the normal form $\Gamma \vdash_{\mathrm{NF}} \mathsf{letmap}_{\mathrm{SL}}\, n\, (\mathit{reify}_{A;(\Gamma,B)}\, x) : \Diamond A$, which is the desired result. The function *reflect*, on the other hand, constructs a value pair of type $[\![\Diamond A]\!]_\Gamma = \Sigma_\Delta.\Gamma \lhd_{\mathrm{SL}} \Delta \times [\![A]\!]_\Delta$ using the given neutral $\Gamma \vdash_{\mathrm{NE}} n : \Diamond A$ and picking Γ, A for the witness Δ to obtain a value of type $[\![A]\!]_{\Gamma,A}$ by reflecting the neutral of $\Gamma, A \vdash_{\mathrm{VAR}} \mathsf{zero} A$. These functions are key to defining *quote* : $(\forall \Delta.\, [\![\Gamma]\!]_\Delta \to [\![A]\!]_\Delta) \to \Gamma \vdash_{\mathrm{NF}} A$, which in turn gives us the function *norm*:

$$\mathit{norm}\, t = \mathit{quote}\, [\![t]\!]$$

NbE models can be constructed likewise for the calculi λ_{SRL}, λ_{SJL} and λ_{LL}. The normal forms of these calculi are defined in Figure 6. To construct the model, we uniformly pick contexts for worlds, the $\leq$ relation for R_i, and the below defined respective modal accessibility relation for R_m. As before, we pick neutrals terms for valuation.

$$\mathsf{nil} : \Gamma \lhd_{\mathrm{SRL}} \Gamma \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A}{\mathsf{single}\, n : \Gamma \lhd_{\mathrm{SRL}} \Gamma, A}$$

$$\frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A}{\mathsf{single}\, n : \Gamma \lhd_{\mathrm{SJL}} \Gamma, A} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \qquad m : \Gamma, A \lhd_{\mathrm{SJL}} \Delta}{\mathsf{cons}\, n\, m : \Gamma \lhd_{\mathrm{SJL}} \Delta}$$

$$\mathsf{nil} : \Gamma \lhd_{\mathrm{LL}} \Gamma \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \qquad m : \Gamma, A \lhd_{\mathrm{LL}} \Delta}{\mathsf{cons}\, n\, m : \Gamma \lhd_{\mathrm{LL}} \Delta}$$

Observe that relation $\lhd_{\mathrm{LL}}$ satisfies the inclusion condition (we can define function *incl*) and is reflexive (we can define refl_m) and transitive (we can define trans_m). On the other hand, the

$$\frac{\Gamma \vdash_{\mathrm{NF}} n : A}{\Gamma \vdash_{\mathrm{NF}} \mathsf{return}_{\mathrm{SRL}}\, n : \Diamond A}\ \text{SRL/NF/}\Diamond\text{-Return} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \quad \Gamma, A \vdash_{\mathrm{NF}} m : B}{\Gamma \vdash_{\mathrm{NF}} \mathsf{letmap}_{\mathrm{SRL}}\, n\, m : \Diamond B}\ \text{SRL/NF/}\Diamond\text{-Letmap}$$

$$\frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \quad \Gamma, A \vdash_{\mathrm{NF}} m : B}{\Gamma \vdash_{\mathrm{NF}} \mathsf{letmap}_{\mathrm{SJL}}\, n\, m : \Diamond B}\ \text{SJL/NF/}\Diamond\text{-Letmap} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \quad \Gamma, A \vdash_{\mathrm{NF}} m : \Diamond B}{\Gamma \vdash_{\mathrm{NF}} \mathsf{let}_{\mathrm{SJL}}\, n\, m : \Diamond B}\ \text{SJL/NF/}\Diamond\text{-Let}$$

$$\frac{\Gamma \vdash_{\mathrm{NF}} n : A}{\Gamma \vdash_{\mathrm{NF}} \mathsf{return}_{\mathrm{LL}}\, n : \Diamond A}\ \text{LL/NF/}\Diamond\text{-Return} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \Diamond A \quad \Gamma, A \vdash_{\mathrm{NF}} m : \Diamond B}{\Gamma \vdash_{\mathrm{NF}} \mathsf{let}_{\mathrm{LL}}\, n\, m : \Diamond B}\ \text{LL/NF/}\Diamond\text{-Let}$$

Figure 6 Normal forms for modal fragments of λ_{SRL}, λ_{SJL} and λ_{LL}

relations $\lhd_{\mathrm{SRL}}$ and $\lhd_{\mathrm{SJL}}$ both satisfy the inclusion condition and are respectively reflexive and transitive, but not the other way round. The main idea behind the definitions of these relations is that they imitate the binding structure of the normal forms in Figure 6.

▶ **Theorem 11** (Correctness of normalization). *For all terms $\Gamma \vdash t : A$ in $\lambda_{SL}/\lambda_{SRL}/\lambda_{SJL}/\lambda_{LL}$, there exists a normal form $\Gamma \vdash_{NF} n : A$ such that $t \sim n$.*

Proof. By virtue of the function *norm*, we get that every term t has a normal form $\mathit{norm}\, t$. Using a standard logical relation based argument we can further show that $t \sim \mathit{norm}\, t$. ◀

▶ **Corollary 12** (Completeness of proof-relevant possible-world semantics). *For any two terms $\Gamma \vdash t, u : A$ in $\lambda_{SL}/\lambda_{SRL}/\lambda_{SJL}/\lambda_{LL}$, if $[\![t]\!] = [\![u]\!]$ in all proof-relevant possible-world models determined by the respective $\lambda_{SL}/\lambda_{SRL}/\lambda_{SJL}/\lambda_{LL}$-frames, then $\Gamma \vdash t \sim u : A$.*

Proof. In the respective NbE model, we know $[\![t]\!] = [\![u]\!]$ implies $\mathit{norm}\, t = \mathit{norm}\, u$ by definition of *norm*. By Theorem 11, we also know $t \sim \mathit{norm}\, t$ and $u \sim \mathit{norm}\, u$, thus $t \sim u$. ◀

▶ **Corollary 13** (Inadmissibility of irrelevant modal axioms). *The axiom R is not derivable in λ_{SL} or λ_{SJL}, and similarly the axiom J is not derivable in λ_{SL} or λ_{SRL}.*

Proof. We first observe that for any neutral term $\Gamma \vdash_{\mathrm{NE}} n : A$, the type A is a subformula of some type in context Γ. We then show by case analysis that there cannot exist a derivation of the judgement $\cdot \vdash_{\mathrm{NF}} A \Rightarrow \Diamond A$ in λ_{SL} or λ_{SJL}, and thus there cannot exist a derivation of axiom R in either calculus—because every term must have a normal form, as shown by the normalization function. A similar argument can be given for axiom J in λ_{SL} and λ_{SRL}. ◀

6 Related and further work

Simpson [35, Chapter 3] gives a comprehensive summary of several IMLs alongside a detailed discussion of their characteristic axioms and possible-world semantics. Notable early work on IMLs can be traced back to Fischer-Servi [22, 34], Božić and Došen [12], Sotirov [36], Plotkin and Stirling [33], Wijesekera [38], and many others since.

Global vs local interpretation. Fairtlough and Mendler [20] give a different presentation of LL. The truth of their lax modality $\bigcirc$ is defined "globally" as follows:

$$\mathcal{M}, w \Vdash \bigcirc A \text{ iff for all } w' \text{ s.t. } w\ R_i\ w' \text{ there exists } v \text{ with } w'\ R_m\ v \text{ and } \mathcal{M}, v \Vdash A$$

Notably, their semantics does not require the forward confluence condition. In the presence of forward confluence, this definition is equivalent to the "local" one we have chosen in Section 2 for the $\lozenge$ modality [35, 18], which means $\bigcirc A$ is true if and only if $\lozenge A$ is true. This observation can also be extended to the respectively determined presheaf functors:

▶ **Proposition 14.** *The presheaf functors $\lozenge$ and $\bigcirc$ are naturally isomorphic.*

In modal logic, the forward confluence condition forces the axiom $\lozenge(A \vee B) \implies \lozenge A \vee \lozenge B$ to be true [6], which may not be desirable in some applications. This observation, however, presupposes that the satisfaction clause for the disjunction connective is defined as follows:

$$\mathcal{M}, w \Vdash A \vee B \text{ iff } \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B$$

This "Kripke-style" interpretation of disjunction is not suitable for our purposes given that our objective is to constructively prove completeness for lambda calculi using possible-world semantics. Completeness in the presence of sum types in lambda calculi is a notorious matter [4, 21] that requires further investigation in the presence of the lax modality.

Box modality in lax logic. Fairtlough and Mendler [20] note that "there is no point" in defining a $\Box$ modality for LL since it "yields nothing new". With the following standard extension of the satisfaction clause for $\Box$:

$$\mathcal{M}, w \Vdash \Box A \text{ iff for all } w' \text{ s.t. } w\ R_i\ w' \text{ and for all } v \text{ s.t. } w'\ R_m\ v \text{ implies } \mathcal{M}, v \Vdash A$$

it follows that $F \models A$ if and only if $F \models \Box A$ for an arbitrary LL-frame F, making $\Box$ a logically meaningless addition to LL.

Proof-relevant semantics Alechina at al. [3] study a connection between categorical and possible-world models of lax logic. They show that an LL-*modal algebra* determines a possible-world model of LL [3, Theorem 4] via the Stone representation, and observe that a modal algebra is a "thin" categorical model, whose morphisms are given by the partial-order relation of the algebra. This connection, while illuminating, does not satisfy an important requirement motivating Mitchell and Moggi's [29] work: to construct models of lambda calculi by leveraging the possible-world semantics of the corresponding logic. Our proof-relevant possible-world semantics satisfies this requirement and is the key to constructing NbE models.

Kavvos [26, 27] develops proof-relevant possible-world semantics (calling it "Two-dimensional Kripke semantics") for the modal logic corresponding to the minimal Fitch-style calculus, which is namely the logic of Galois connections due to Dzik et al [19]. Kavvos adopts a categorical perspective and shows that profunctors determine an adjunction on presheaves, which can be used to model $\blacklozenge \dashv \Box$. Kavvos' profunctor condition is the proof-relevant refinement of Sotirov's [36] bimodule frame condition which states that $R_i ; R_m ; R_i \subseteq R_m$

Proof-relevant possible-world semantics and its connection to NbE for modal lambda calculi is a novel consideration in our work. Valliappan et al [37] prove normalization for Fitch-style modal lambda calculi [11, 13], consisting of the necessity modality $\Box$ and its left adjoint $\blacklozenge$ using possible-world semantics with a proof-*irrelevant* relation R_m.

Frame correspondence. The study of necessary and sufficient frame conditions for modal axioms, known as *frame correspondence*, appears to be tricky in the proof-relevant setting. Plotkin and Stirling [33] prove a remarkably general correspondence theorem (Theorem 2.1) that tells us that the reflexivity of $R_m ; {R_i}^{-1}$ corresponds to axiom R and ${R_m}^2 \subseteq R_m ; {R_i}^{-1}$ corresponds to axiom J. We have not studied frame correspondence in this article, but leave it as a matter for future work. The profunctor perspective of Kavvos [26] might help here.

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A Definitions of strong functors

A *strong* functor $F : \mathcal{C} \to \mathcal{C}$ for a cartesian category $\mathcal{C}$ is an endofunctor on $\mathcal{C}$ with a natural transformation $\theta_{P,Q} : P \times FQ \to F(P \times Q)$ natural in $\mathcal{C}$-objects P and Q such that the following diagrams stating coherence conditions commute:

$$\begin{array}{ccc} \mathit{1} \times FP & \xrightarrow{\theta_{\mathit{1},P}} & F(\mathit{1} \times P) \\ & {\scriptstyle \pi_2} \searrow \quad \swarrow {\scriptstyle F\pi_2} & \\ & FP & \end{array}$$

$$\begin{array}{ccccc} (P \times Q) \times FR & & \xrightarrow{\theta_{P \times Q,R}} & & F((P \times Q) \times R) \\ \downarrow {\scriptstyle \alpha_{P,Q,FR}} & & & & \downarrow {\scriptstyle F\alpha_{P,Q,R}} \\ P \times (Q \times FR) & \xrightarrow{\mathit{id}_P \times \theta_{Q,R}} & P \times F(Q \times R) & \xrightarrow{\theta_{P,Q \times R}} & F(P \times (Q \times R)) \end{array}$$

Observe that the terminal object $\mathit{1}$, the projection morphism $\pi_2 : P \times Q \to Q$ and the associator morphism $\alpha_{P,Q,R} : (P \times Q) \times R \to P \times (Q \times R)$ (for all $\mathcal{C}$-objects P, Q, R) live in the cartesian category $\mathcal{C}$.

A *pointed* functor $F : \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$ is an endofunctor on $\mathcal{C}$ equipped with a natural transformation $\mathit{point} : \mathit{Id} \dot{\to} F$ from the identity functor Id on $\mathcal{C}$.

A strong and pointed functor F is said to be *strong pointed*, when it satisfies an additional coherence condition that point is a strong natural transformation, meaning that the following diagram stating a coherence condition commutes:

$$\begin{array}{ccc} & P \times Q & \\ & {\scriptstyle \mathit{id}_P \times \mathit{point}_Q} \swarrow \quad \searrow {\scriptstyle \mathit{point}_{P \times Q}} & \\ P \times FQ & \xrightarrow{\theta_{P,Q}} & F(P \times Q) \end{array}$$

A *semimonad* $F : \mathcal{C} \to \mathcal{C}$, or *joinable* functor, on a category $\mathcal{C}$ is an endofunctor on $\mathcal{C}$ that forms a semigroup in the sense that it is equipped with a "multiplication" natural transformation $\mu : F^2 \dot{\to} F$ that is "associative" as $\mu_P \circ \mu_{FP} = \mu_P \circ F(\mu_P) : F^3 P \to FP$.

A strong functor F that is also a semimonad is a *strong semimonad* when μ is a strong natural transformation, meaning that the following coherence condition diagram commutes:

$$\begin{array}{ccccc} P \times FFQ & \xrightarrow{\theta_{P,FQ}} & F(P \times FQ) & \xrightarrow{F\theta_{P,Q}} & FF(P \times Q) \\ \downarrow{\scriptstyle \mathit{id}_P \times \mu_Q} & & & & \downarrow{\scriptstyle \mu_{P \times Q}} \\ P \times FQ & & \xrightarrow{\theta_{P,Q}} & & F(P \times Q) \end{array}$$

A strong functor F that is also a semimonad is a *strong semimonad* when μ is a strong natural transformation, meaning that the following coherence condition diagram commutes:

$$\begin{array}{ccccc} P \times FFQ & \xrightarrow{\theta_{P,FQ}} & F(P \times FQ) & \xrightarrow{F\theta_{P,Q}} & FF(P \times Q) \\ \downarrow{\scriptstyle \mathit{id}_P \times \mu_Q} & & & & \downarrow{\scriptstyle \mu_{P \times Q}} \\ P \times FQ & & \xrightarrow{\theta_{P,Q}} & & F(P \times Q) \end{array}$$

A *monad* $F : \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$ is a semimonad that is pointed, such that the natural transformation $\mathit{point} : \mathit{Id} \dot{\to} F$ is the left and right unit of multiplication $\mu : F^2 \dot{\to} F$ in the sense that $\mu_P \circ F\mathit{point}_P = \mathit{id}_{FP}$ and $\mu_P \circ \mathit{point}_{FP} = \mathit{id}_{FP}$ for some $\mathcal{C}$-object P.

A strong functor F that is also a monad is a *strong monad* when the natural transformations point and μ of the monad are both strong natural transformations, making F both a strong pointed functor and a strong semimonad.