Strong Functors as Modalities

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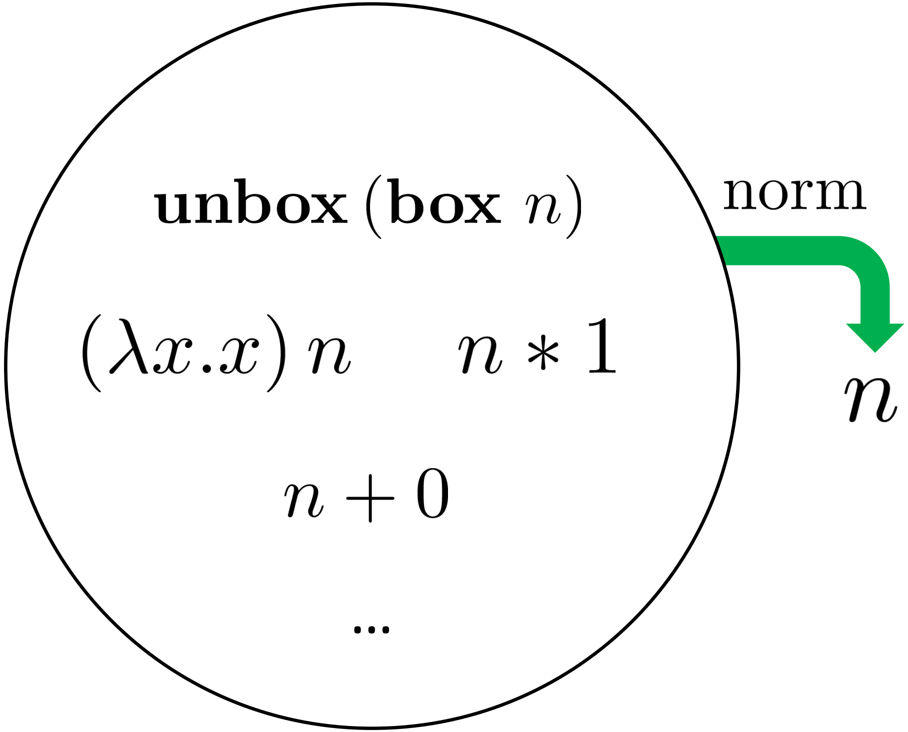
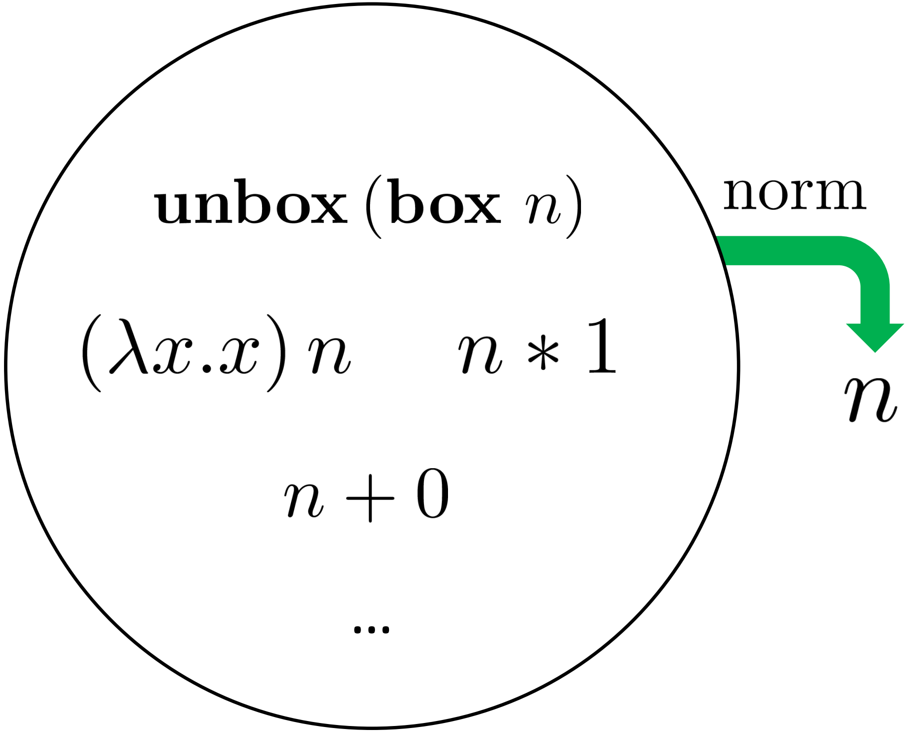
Normalization is important

Why normalization matters



Why normalization matters



- Canonicity
- Conversion checking
- Completeness
- Noninterference
- Optimization
- ...

Normalization is hard

Normalization is monolithic

How do we break it down?

Normalization by Evaluation (NbE)

$$
\begin{aligned}
\text{eval} \quad &: \Gamma \vdash A \to (\llbracket \Gamma \rrbracket \Rightarrow \llbracket A \rrbracket) \\
\text{quote} &: (\llbracket \Gamma \rrbracket \Rightarrow \llbracket A \rrbracket) \to \Gamma \vdash_{\text{NF}} A
\end{aligned}
$$

$$
\begin{aligned}
&\text{norm} : \Gamma \vdash A \to \Gamma \vdash_{\text{NF}} A \\
&\text{norm} = \text{quote} \circ \text{eval}
\end{aligned}
$$

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- Where do we begin?
- How do we prove it correct (“sound”)?
- How do we modify it when the language changes?

Overarching goal

Construct NbE model for feature X using modules that are reusable for several languages that exhibit X

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Strong functors

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Lambda calculi

The question we are really asking

$$\text{What is } [\![\Gamma]\!] \Rightarrow [\![A]\!]?$$

Possible-world (or ”Kripke”) semantics to the rescue

$$\begin{aligned} \text{eval} &: \Gamma \vdash A \to (\forall w.\ [\![\Gamma]\!]_w \to [\![A]\!]_w) \\ \text{quote} &: (\forall w.\ [\![\Gamma]\!]_w \to [\![A]\!]_w) \to \Gamma \vdash_{\text{NF}} A \end{aligned}$$

$$\begin{aligned} &\text{norm} : \Gamma \vdash A \to \Gamma \vdash_{\text{NF}} A \\ &\text{norm} = \text{quote} \circ \text{eval} \end{aligned}$$

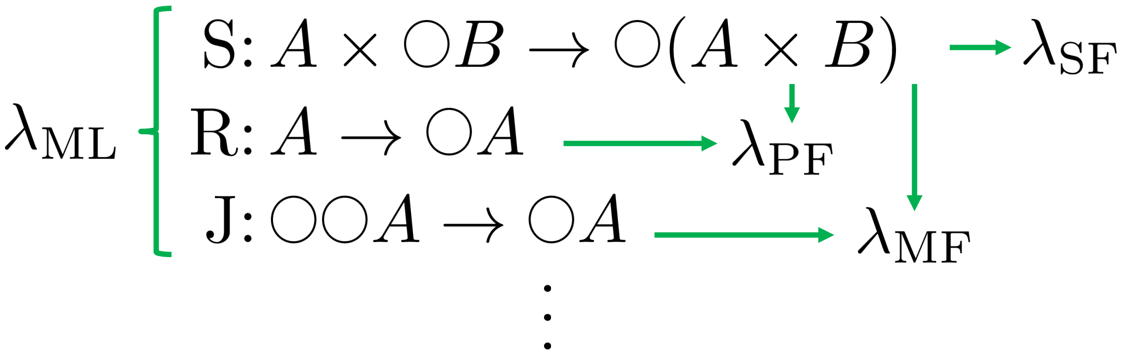
Strong functors as modalities

$$\bigcirc : \mathrm{Type} \to \mathrm{Type}$$

Strong functors in play

$$\mathrm{S}\colon A \times \bigcirc B \to \bigcirc(A \times B)$$
$$\mathrm{R}\colon A \to \bigcirc A$$
$$\mathrm{J}\colon \bigcirc\bigcirc A \to \bigcirc A$$
$$\vdots$$

Strong functors in play



Key insights

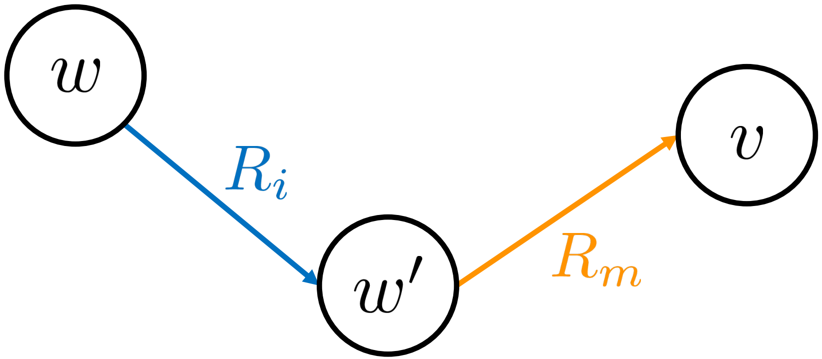
1] NbE using possible-world semantics for monads

2] Decomposition of possible-world models of monads

Possible-world semantics

Frame: $(\mathcal{W}, R_i, R_m)$

R_i reflexive, transitive



Possible-world semantics

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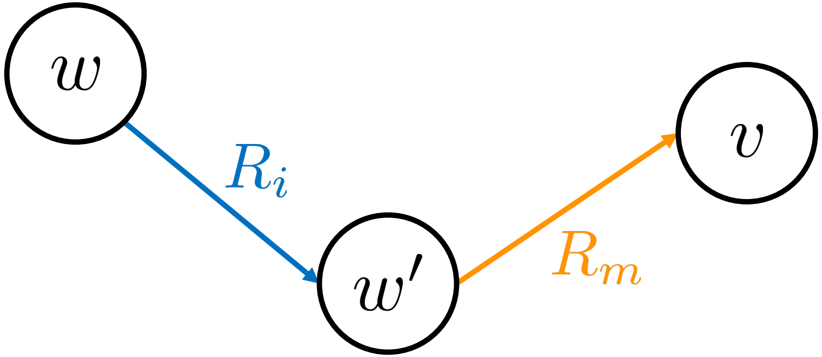

$\ldots$

$$[\![\bigcirc A]\!]_w = \forall w'.\, w\ R_i\ w' \to \exists v.\, w'\ R_m\ v \times [\![A]\!]_v$$

Moggi's monadic metalanguage

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathrm{return}\ t : \bigcirc A} \qquad \frac{\Gamma \vdash t : \bigcirc A \quad \Gamma, A \vdash u : \bigcirc B}{\Gamma \vdash \mathrm{let}_{\mathrm{ML}}\ t\ u : \bigcirc B}$$

Moggi's monadic metalanguage

$$\Gamma \vdash_{\text{NF}} \mathsf{let}_{\text{ML}}\, n_1\, (\mathsf{let}_{\text{ML}}\, n_2\, \ldots (\mathsf{let}_{\text{ML}}\, n_j\, (\mathsf{return}_{\text{ML}}\, m)))...) : \bigcirc B$$

Moggi's monadic metalanguage

$$\Gamma, A_{i-1} \vdash_{\mathrm{NE}} n_i : \bigcirc A_i$$

$$\Gamma \vdash_{\mathrm{NF}} \mathsf{let}_{\mathrm{ML}}\, n_1\, (\mathsf{let}_{\mathrm{ML}}\, n_2\, \ldots (\mathsf{let}_{\mathrm{ML}}\, n_j\, (\mathsf{return}_{\mathrm{ML}}\, m)))\ldots) : \bigcirc B$$

Moggi's monadic metalanguage

$$\Gamma, A_{i-1} \vdash_{\mathrm{NE}} n_i : \bigcirc A_i \qquad \Gamma, A_1, \ldots, A_j \vdash_{\mathrm{NF}} m : B$$

$$\Gamma \vdash_{\mathrm{NF}} \mathsf{let}_{\mathrm{ML}}\, n_1\, (\mathsf{let}_{\mathrm{ML}}\, n_2\, \ldots (\mathsf{let}_{\mathrm{ML}}\, n_j\, (\mathsf{return}_{\mathrm{ML}}\, m)))\ldots) : \bigcirc B$$

Constructing an NbE model for monads

Frame $\mathcal{F}$: $(\mathcal{W}, R_i, R_m)$

R_i, R_m reflexive, transitive

$R_m \subseteq R_i$

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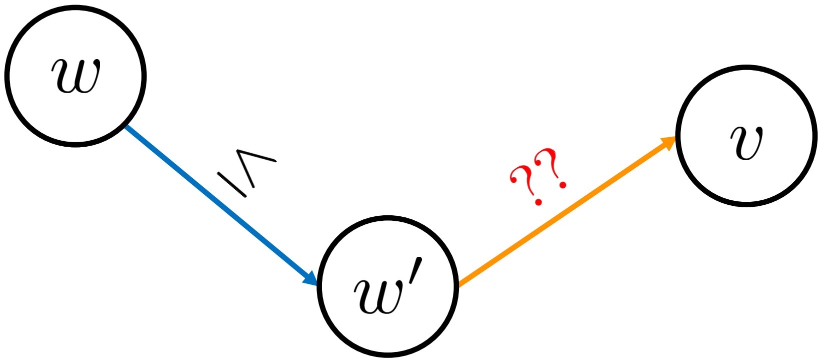
$\mathcal{W} = \text{Contexts}$

$R_i = \leq \text{(OPEs)}$

$R_m = ??$



Defining a modal accessibility relation

$$\mathsf{nil} : \Gamma \lhd_{\mathrm{ML}} \Gamma \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \bigcirc A \qquad e : \Gamma, A \lhd_{\mathrm{ML}} \Delta}{\mathsf{cons}\, n\, e : \Gamma \lhd_{\mathrm{ML}} \Delta}$$

$$\mathsf{cons}\, n_1\, (\mathsf{cons}\, n_2\, (\ldots\ldots (\mathsf{cons}\, n_j\, \mathsf{nil}))\ldots) : \Gamma \lhd_{\mathrm{ML}} \Gamma, A_1, A_2, \ldots A_j$$

$\lhd_{\mathrm{ML}}$ is reflexive, transitive, and included in $\leq$

Reification

$$\textit{reify}_{A;\Gamma} : [\![A]\!]_\Gamma \to \Gamma \vdash_{\text{NF}} A$$

$$\textit{reify}_{\bigcirc A;\Gamma}\, f = \textit{reifyAcc}\; e\, (\mathsf{return}_{\text{ML}}\, (\textit{reify}_{A;\Delta} v))$$

$$\quad \text{where } (e : \Gamma \lhd_{\text{ML}} \Delta, v : [\![A]\!]_\Delta) = f\; \mathsf{id}_{\leq}$$

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$$\mathit{reifyAcc}_{\Gamma;\Delta} : \Gamma \lhd_{\mathrm{ML}} \Delta \to (\Delta \vdash_{\mathrm{NF}} \bigcirc A \to \Gamma \vdash_{\mathrm{NF}} \bigcirc A)$$

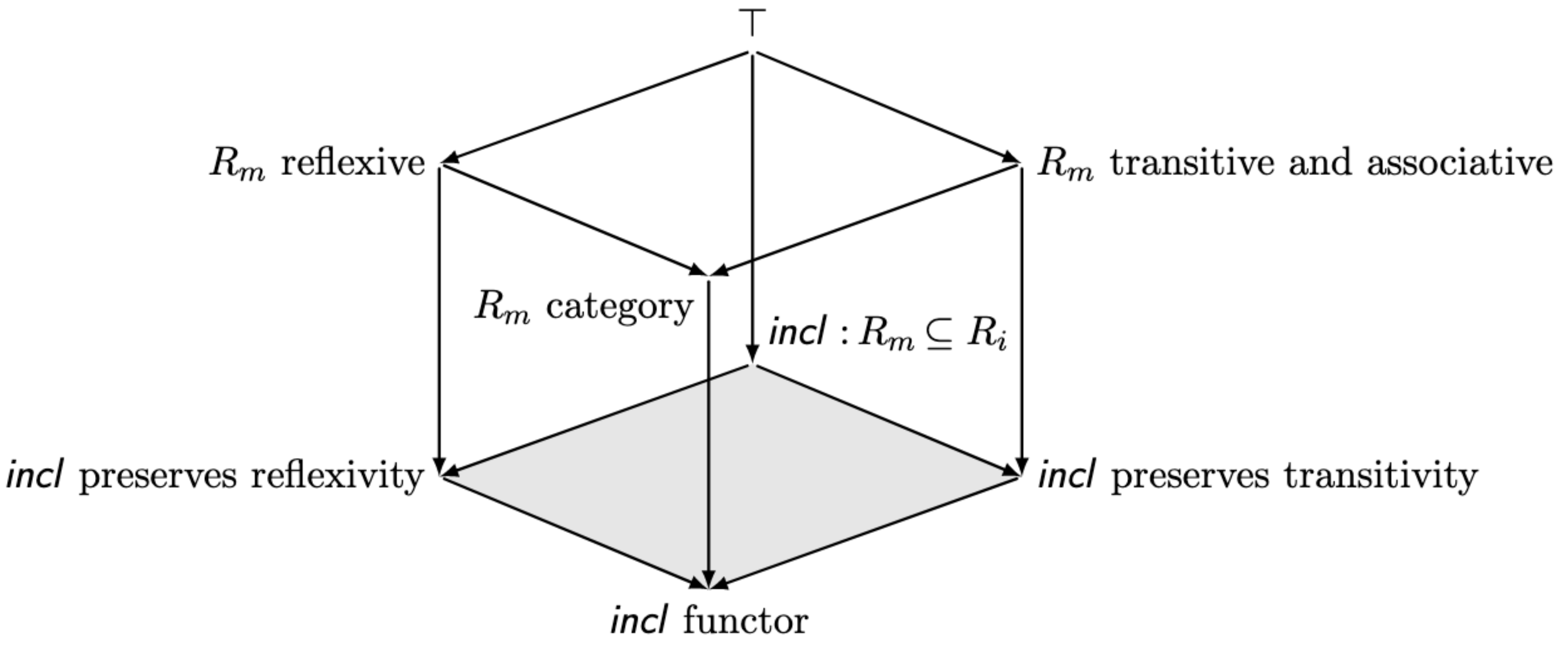
$$\mathit{reifyAcc}_{\Gamma;\Gamma}\ \mathsf{nil} = id$$

$$\mathit{reifyAcc}_{\Gamma;\Delta}\, (\mathsf{cons}\, (n : \Gamma \vdash_{\mathrm{NE}} \bigcirc B)\, e) = (\lambda m.\, \mathsf{let}_{\mathrm{ML}}\, n\, m) \circ (\mathit{reifyAcc}_{(\Gamma,B);\Delta}\, e)$$

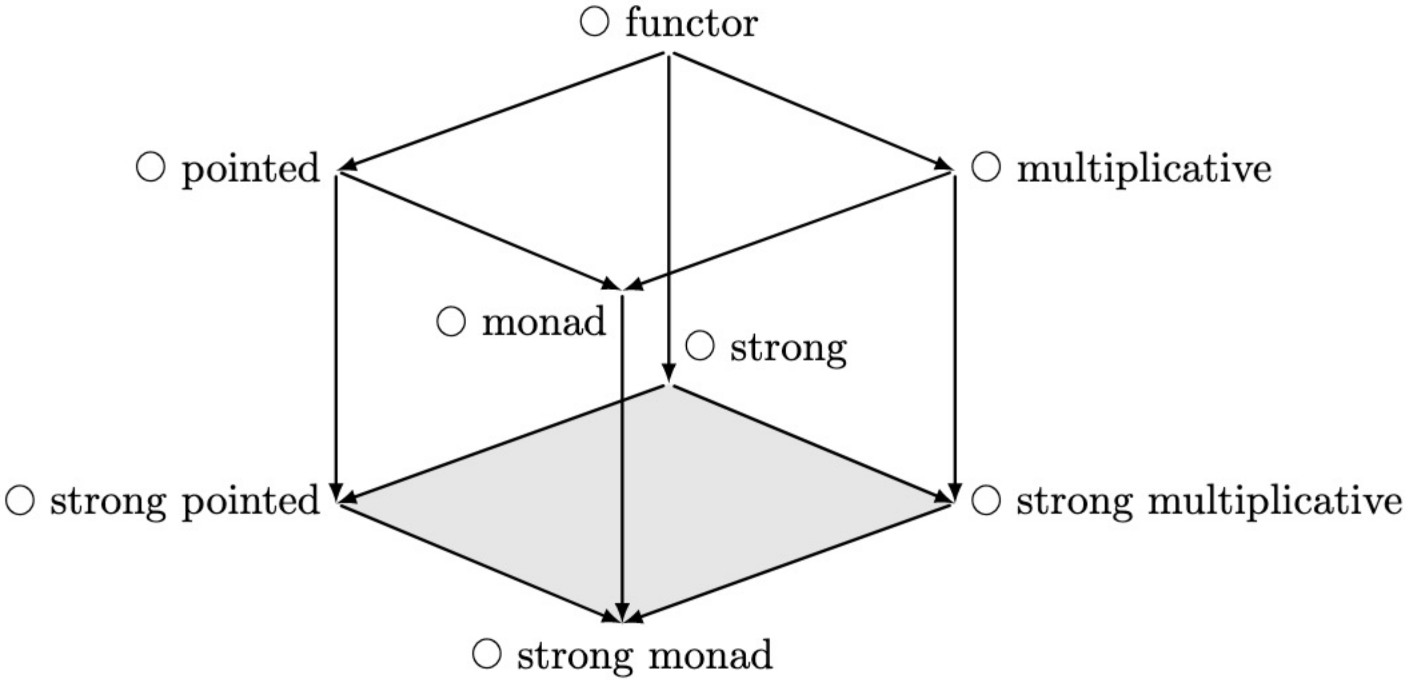
Reification

$$(\mathsf{cons}\, n_1\, (\ldots (\mathsf{cons}\, n_j\, \mathsf{nil})\ldots), v) \quad \rightsquigarrow \quad \mathsf{let}_{\mathrm{ML}}\, n_1\, (\ldots (\mathsf{let}_{\mathrm{ML}}\, n_j\, (\mathsf{return}_{\mathrm{ML}}\, (\textit{reify}_{A;\Delta}\, v)))\ldots)$$

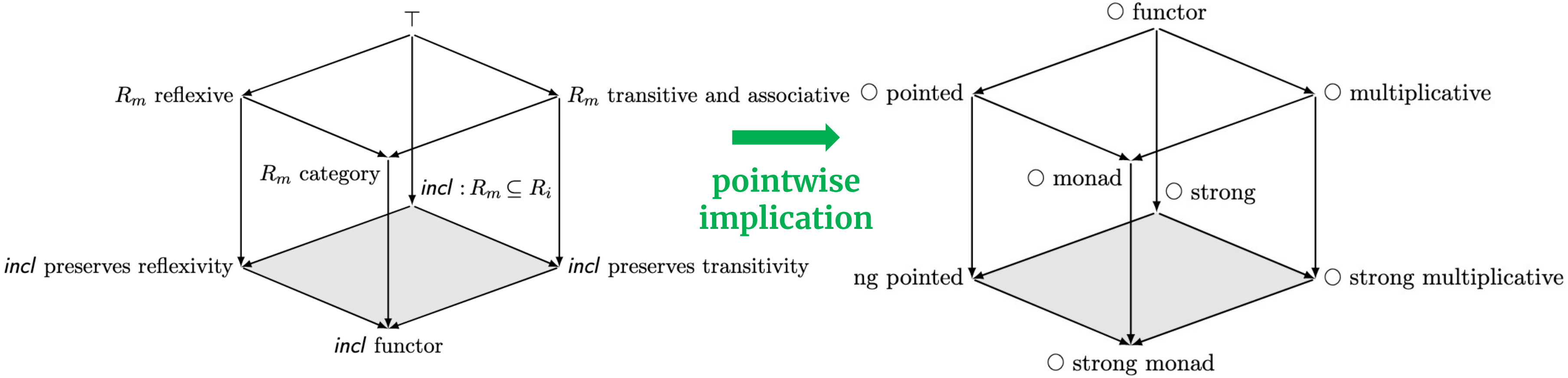
Decomposition of frame conditions



Decomposition of a strong monad



Main theorem



Calculus for strong functors

$$\textsc{SF}/\bigcirc\text{-}\textsc{Letmap}\quad \frac{\Gamma \vdash t : \bigcirc A \qquad \Gamma, A \vdash u : B}{\Gamma \vdash \mathsf{letmap}_{\mathrm{SF}}\, t\, u : \bigcirc B}$$

$$\textsc{SF}/\bigcirc\text{-}\textsc{Letmap-Id}\quad \frac{\Gamma \vdash t : \bigcirc A}{\Gamma \vdash t \sim \mathsf{letmap}_{\mathrm{SF}}\, t\, (\mathsf{var\ zero}) : \bigcirc A}$$

$$\textsc{SF}/\bigcirc\text{-}\textsc{Letmap-Comp}\quad \frac{\Gamma \vdash t : \bigcirc A \qquad \Gamma, A \vdash u : B \qquad \Gamma, B \vdash u' : C}{\Gamma \vdash \mathsf{letmap}_{\mathrm{SF}}\, (\mathsf{letmap}_{\mathrm{SF}}\, t\, u)\, u' \sim \mathsf{letmap}_{\mathrm{SF}}\, t\, (u'[u]) : \bigcirc C}$$

NbE for strong functors

$$\frac{\Gamma \vdash_{\text{NE}} n : \bigcirc A \qquad \Gamma, A \vdash_{\text{NF}} m : B}{\Gamma \vdash_{\text{NF}} \mathsf{letmap}_{\text{SF}}\, n\, m : \bigcirc B}\ \text{SF/NF/}\bigcirc\text{-L}\textsc{etmap}$$

$$\frac{\Gamma \vdash_{\text{NE}} n : \bigcirc A}{\mathsf{single}\, n : \Gamma \lhd_{\text{SF}} \Gamma, A}$$

Reification for strong functors

$$\textit{reify}_{\bigcirc A;\Gamma}\, f = \mathsf{letmap}_{\mathrm{SF}}\, n\, (\textit{reify}_{A;\Gamma,B}\, v))$$
$$\quad \text{where } ((\mathsf{single}\, n : \Gamma \lhd_{\mathrm{SF}} \Gamma, B), v : [\![A]\!]_\Delta) = f\, \mathsf{id}_{\leq}$$

NbE for pointed and multiplicative functors

$$\mathsf{nil} : \Gamma \lhd_{\mathrm{PF}} \Gamma \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \bigcirc A}{\mathsf{single}\, n : \Gamma \lhd_{\mathrm{PF}} \Gamma, A}$$

$$\frac{\Gamma \vdash_{\mathrm{NE}} n : \bigcirc A}{\mathsf{single}\, n : \Gamma \lhd_{\mathrm{MF}} \Gamma, A} \qquad \frac{\Gamma \vdash_{\mathrm{NE}} n : \bigcirc A \qquad e : \Gamma, A \lhd_{\mathrm{MF}} \Delta}{\mathsf{cons}\, n\, e : \Gamma \lhd_{\mathrm{MF}} \Delta}$$

Limitations and future work

1] Need general way to show *quote* is a left inverse

2] Sums require intervention, perhaps we need:

$$[\![\bigcirc A]\!]_w = \forall w'.\, w\ R_i\ w' \to \exists(\mathcal{C} : \mathsf{Cover}_{R_m}\ w').\, \forall v \in \mathcal{C}.\ \to [\![A]\!]_v$$

In a nutshell

Normalization for strong functors be achieved in a modular fashion by constructing NbE models as instances of their possible-world semantics.

Mechanization: github.com/nachivpn/fam

EOM